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Uniqueness of Axisymmetric Viscous Flows Originating from Positive Linear Combinations of Circular Vortex Filaments

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Abstract. Following the recent papers by Gallay and Šverák (Conflu Math 7:67–92, 2015; Uniqueness of axisymmetric viscous flows originating from circular vortex filaments, arXiv:1609.02030), in the line of work initiated by Feng and Šverák (Arch Ration Mech Anal 215: 89–123 2015), we prove the uniqueness of a solution of the axisymmetric Navier–Stokes equations without swirl when the initial vorticity is a linear combination of positive Dirac masses.

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1. Introduction

In 3-D incompressible ideal fluids, a vortex ring is an axisymmetric flow whose vorticity is entirely concentrated in a solid torus, which moves with constant speed along the symmetry axis. See [1,4–6] for the existence of vortex ring solutions to the 3-D Euler equations.

However, for viscous fluids, the vortex ring solutions cannot exist, since all localized structures will be spread out by diffusion. Thus it is natural to consider the Navier–Stokes equations with a vortex filament, and more generally with positive linear combinations of circular vortex filaments which have a common axis of symmetry as initial data.

To state this precisely, let us start with the Navier–Stokes equations in $\mathbb{R}^3$

$$\partial_t u + u\cdot\nabla u - \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0, \qquad (t,y)\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.1}$$

where $u(t,y) = (u^1,u^2,u^3)$ stands for the velocity field and p the scalar pressure function of the fluid, which guarantees that the velocity field remains divergence free.

In the following, we restrict ourselves to the axisymmetric solutions without swirl of (1.1), for which the velocity field u and its vorticity $\omega \stackrel{\text{def}}{=} \operatorname{curl} u$ take the particular form

$$u(t,y) = u^r(t,r,z)e_r + u^z(t,r,z)e_z, \quad \omega(t,y) = \omega^\theta(t,r,z)e_\theta,$$

where (r,θ,z) denotes the cylindrical coordinates in $\mathbb{R}^3$ so that $y = (r\cos\theta, r\sin\theta, z)$, and

$$e_r = (\cos\theta,\sin\theta,0),\ e_\theta = (-\sin\theta,\cos\theta,0),\ e_z = (0,0,1),\ r = \sqrt{x_1^2+x_2^2}.$$

As in [9], we equip the half-plane $\Omega = \{(r,z)|r>0, z\in\mathbb{R}\}$ with the measure $drdz$. More precisely, for any measurable function $f:\Omega\to\mathbb{R}$, we denote

$$\|f\|_{L^p(\Omega)} \stackrel{\text{def}}{=} \Big(\int_\Omega |f(r,z)|^p drdz\Big)^{\frac{1}{p}} < \infty,\ 1\le p<\infty,$$

and $\|f\|_{L^\infty(\Omega)}$ to be the essential supremum of $|f|$ on Ω. For notational simplicity, we shall always denote a generic point in Ω by $x = (r,z)$.

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Recalling the axisymmetric Biot-Savart law discussed in Section 2 of [9], we know that for any given $\omega^\theta \in L^1(\Omega) \cap L^\infty(\Omega)$ which vanishes on $r = 0$, the linear elliptic system

$$\begin{cases} \partial_r u^r + \frac{1}{r} u^r + \partial_z u^z = 0, \quad \partial_z u^r - \partial_r u^z = \omega^\theta, \quad \text{on } \Omega, \\ u^r|_{r=0} = 0, \quad \partial_r u^z|_{r=0} = 0, \end{cases}$$

has a unique solution $(u^r, u^z) \in C(\Omega)^2$ which vanishes at infinity. We denote this solution by $u = BS[\omega^\theta]$. Hence we only need to study the equation for ω^θ:

$$\partial_t \omega^\theta + (u^r \partial_r + u^z \partial_z)\omega^\theta - \frac{u^r \omega^\theta}{r} = \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} \right) \omega^\theta. \tag{1.2}$$

Now let us discuss the initial condition. We first recall from [9] that, the axisymmetric vorticity Eq. (1.2) is globally well-posed whenever the initial vorticity is in $L^1(\Omega)$. As a natural extension, they then considered the case of an initial vorticity in $\mathcal{M}(\Omega)$, which denotes the set of all real-valued finite regular measures on Ω, equipped with the total variation norm

$$\|\mu\|_{\mathrm{tv}} \stackrel{\mathrm{def}}{=} \sup \left\{ \int_\Omega \phi \, d\mu \,\middle|\, \phi \in C_0(\Omega), \|\phi\|_{L^\infty(\Omega)} \le 1 \right\},$$

where $C_0(\Omega)$ denotes the set of all real-valued continuous functions on Ω that vanishes at infinity and on the boundary $\partial\Omega$. It is also proved in [9] that (1.2) is globally well-posed if the initial vorticity μ is in $\mathcal{M}(\Omega)$ with a small enough atomic part.

As mentioned in the second paragraph, we focus here on the particular case

$$\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}, \tag{1.3}$$

where n is some positive integer, α_i is some positive constant, and δ_{x_i} is the Dirac mass at point $x_i = (r_i, z_i) \in \Omega$ with $r_i > 0$. Such a μ is purely atomic, and we can deduce from [9] that (1.2) is globally well-posed provided that

$$\|\mu\|_{\mathrm{tv}} = \sum_{i=1}^n \alpha_i$$

is small enough. On the other hand, for arbitrary positive values of α_i, [3] gives the existence of a global mild solution, and [10] proves the uniqueness when $n = 1$. In this paper, we are going to prove the uniqueness for general n. Our result states as follows:

Theorem 1.1. *If the initial vorticity μ of (1.2) is given by (1.3), then (1.2) has a unique global mild solution (see Definition 2.1 below) ω^θ in $\mathcal{C}\big(]0,\infty[, L^1(\Omega) \cap L^\infty(\Omega)\big)$, satisfying*

$$\sup_{t>0} \|\omega^\theta(t)\|_{L^1(\Omega)} < \infty, \quad \text{and} \quad \omega^\theta(t) drdz \rightharpoonup \mu \quad \text{as} \quad t \to 0. \tag{1.4}$$

Moreover, there exists some constant C_0 depending only on $(\alpha_i, x_i)_{i=1}^n$, such that whenever $\sqrt{t} \le \frac{1}{2} \min_{1\le i<j\le n}\big\{|x_i - x_j|, r_i\big\}$, there holds the following short time estimate:

$$\Big\| \omega^\theta(t,\cdot) - \frac{1}{4\pi t} \sum_{i=1}^n \alpha_i e^{-\frac{|\cdot - x_i|^2}{4t}} \Big\|_{L^1(\Omega)} \le C_0 \sqrt{t} |\ln t|. \tag{1.5}$$

Let us end up this section with some notations. We use C (resp. C_0) to denote some absolute positive constant (resp. some positive constant depending on $(\alpha_i, x_i)_{i=1}^n$), which may be different in each occurrence. $f \lesssim g$ means that $f \le Cg$ for some constant C. For a Banach space B, we shall use the shorthand $\|u\|_{L^p_T(B)}$ for the norm $\big\| \|u(t,\cdot)\|_B \big\|_{L^p(0,T)}$.

2. Decomposition of the Solution

Following the ideas in [10], where uniqueness has been proved under the assumption that thei nitial vorticity in a single Dirac mass, a natural idea is to decompose the solution into n parts, according to the decomposition of the initial vorticity (1.3). This idea is however nontrivial to implement, due to the nonlinearity of the Eq. (1.2). The strategy is to view the equation on ω^θ as a linear advection-diffusion one, with u being given and to study the properties of its fundamental solution. This will be done in the first subsection.

The purpose of the second subsection will be to show that, at least for short times, ω_i^θ is very close—in the $L^1(\Omega)$ sense—to the Oseen vortex located at x_i with circulation α_i. This goal will be achieved using self-similar variables around the point x_i.

2.1. The Linear Semigroup and the Trace of the Solution at Initial Time

Let us consider the linearized system of (1.2), namely

$$\begin{cases} \partial_t\omega^\theta - \big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\big)\omega^\theta = 0, \quad (t,r,z) \in \mathbb{R}^+ \times \Omega, \\ \omega^\theta|_{r=0} = 0, \quad \omega^\theta|_{t=0} = \omega_0^\theta. \end{cases} \tag{2.1}$$

Denote by $\big(S(t)\big)_{t\geq 0}$ the evolution semigroup determined by this linearized system, which has the following explicit formula

$$\big(S(t)\omega_0^\theta\big)(r,z) = \frac{1}{4\pi t}\int_\Omega \frac{\bar{r}^{\frac12}}{r^{\frac12}} H\left(\frac{t}{r\bar{r}}\right)\exp\left(-\frac{(r-\bar{r})^2 + (z-\bar{z})^2}{4t}\right)\omega_0^\theta(\bar{r},\bar{z})\,d\bar{r}d\bar{z}, \tag{2.2}$$

where $H:(0,\infty)\to\mathbb{R}$ is a smooth function satisfying the following property: $\tau^\alpha H(\tau)$ and $\tau^\beta H'(\tau)$ are bounded on $(0,\infty)$ if $0\leq\alpha\leq\frac32$ and $0\leq\beta\leq\frac52$. One may check Section 3 of [9] for a detailed study of this semigroup.

By using $\big(S(t)\big)_{t\geq 0}$, we can define the mild solutions of (1.2) in the following way:

Definition 2.1. Let $T>0$, we say that $\omega^\theta \in C\big(]0,T[, L^1(\Omega)\cap L^\infty(\Omega)\big)$ is a mild solution of (1.2) on $]0,T[$, if for any $0<t_0<t<T$, there holds the following integral equation

$$\omega^\theta(t) = S(t-t_0)\omega^\theta(t_0) - \int_{t_0}^t S(t-s)\,\mathrm{div}_*\big(u(s)\omega^\theta(s)\big)\,ds. \tag{2.3}$$

Here $u = \mathrm{BS}[\omega^\theta]$ and $\mathrm{div}_*\big(u\omega^\theta\big) \stackrel{\text{def}}{=} \partial_r(u^r\omega^\theta) + \partial_z(u^z\omega^\theta)$.

Before proceeding further, let us recall some *a priori* estimates for the mild solution.

Lemma 2.1. *Let ω^θ be a mild solution of (1.2) on $(0,T)$ satisfying (1.4), $u=\mathrm{BS}[\omega^\theta]$. It is shown in Estimates (2.13), (2.14) of [10] that, for any $t\in]0,T[$, and any $k,\ell\in\mathbb{N}$, there holds*

$$t^{k+\frac{\ell}{2}+\frac12}\|\partial_t^k\nabla_x^\ell u(t)\|_{L^\infty(\Omega)} + t^{\frac32}\|\nabla\omega^\theta(t)\|_{L^\infty(\Omega)} \leq C_0. \tag{2.4}$$

Combining the conclusions of Corollary 2.9, 2.10 and Remark 2.11 in [10], we prove the following.

Proposition 2.1. *For any $T>0$, if $\omega^\theta\in C\big((0,T), L^1(\Omega)\cap L^\infty(\Omega)\big)$ is a mild solution of (1.2) on $(0,T)$ satisfying (1.4), then for any $t\in(0,T)$ and $(r,z)\in\Omega$, we have*

$$\omega^\theta(t,r,z)\geq 0, \quad \|\omega^\theta(t)\|_{L^1(\Omega)} \leq \|\mu\|_{\mathrm{tv}} \quad \text{and} \quad \lim_{t\to 0}\|\omega^\theta(t)\|_{L^1(\Omega)} = \|\mu\|_{\mathrm{tv}}. \tag{2.5}$$

Moreover, for any bounded and continuous function ϕ on Ω, there holds the convergence

$$\int_\Omega \phi(r,z)\omega^\theta(t,r,z)\,drdz \to \int_\Omega \phi\,d\mu, \quad \text{as} \quad t\to 0. \tag{2.6}$$

Noting that although the initial measure μ is no longer a single Dirac mass as considered in [10], it is still supported in $\big[\min_{1\le i\le n} r_i, \max_{1\le i\le n} r_i\big]\times\mathbb{R}$. Thus the estimates of Proposition 3.1, 3.3 and then Lemma 3.8 in [10] still hold for the case here. Precisely, we have

$$\int_0^T \|u^r(t)/r\|_{L^\infty(\Omega)}\,dt \le C_0. \tag{2.7}$$

Next, let us state a particular case of Aronson's pioneering work [2] on the fundamental solution of parabolic equations, which will be a key ingredient in our decomposition.

Proposition 2.2 (Proposition 3.9 of [10]). *Assume that $U,\ V:(0,T)\times\mathbb{R}^3\to\mathbb{R}^3$ are continuous functions such that* $\operatorname{div} U(t,\cdot)=0$, *for all $t\in(0,T)$ and*

$$\sup_{0<t<T} t^{\frac12}\|U(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} = K_1<\infty, \quad \int_0^T \|V(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}\,dt = K_2<\infty.$$

Then the regular solutions of the following type advection-diffusion equation

$$\partial_t f + U\cdot\nabla f - Vf = \Delta f, \quad x\in\mathbb{R}^3, \quad t\in(0,T), \tag{2.8}$$

can be represented in the following way:

$$f(t,x)=\int_{\mathbb{R}^3}\Phi_{U,V}(t,x;s,y)f(s,y)\,dy, \quad x\in\mathbb{R}^3,\ 0<s<t<T,$$

where $\Phi_{U,V}$ is the (uniquely defined) fundamental solution, which satisfies for all $x,y\in\mathbb{R}^3$ and $0<s<t<T$

$$0<\Phi_{U,V}(t,x;s,y)\le \frac{C}{(t-s)^{\frac32}}\exp\Big(-\frac{|x-y|^2}{4(t-s)}+K_1\frac{|x-y|}{\sqrt{t-s}}+K_2\Big). \tag{2.9}$$

It is easy to derive the evolution equation for $\omega=\omega^\theta(t,r,z)e_\theta$ from (1.1) that

$$\partial_t\omega + u\cdot\nabla\omega - r^{-1}u^r\omega = \Delta\omega, \quad x\in\mathbb{R}^3, \quad t\in(0,T), \tag{2.10}$$

which is exactly of the form (2.8) with $U=u,\ V=r^{-1}u^r$. In view of (2.4) and (2.7), the conditions of Proposition 2.2 are satisfied. Thus this ω can be represented as

$$\omega(t,x)=\int_{\mathbb{R}^3}\Phi(t,x;s,y)\omega(s,y)\,dy, \quad x\in\mathbb{R}^3,\ 0<s<t<T.$$

We denoted above by Φ the fundamental solution $\Phi_{U,V}$ where $U=u,\ V=r^{-1}u^r$. From which, we can deduce that ω^θ satisfies

$$\omega^\theta(t,r,z)=\int_\Omega \widetilde{\Phi}(t,r,z;s,r',z')\omega^\theta(s,r',z')\,dr'dz', \quad 0<s<t<T, \tag{2.11}$$

where

$$\widetilde{\Phi}(t,r,z;s,r',z')=\int_{-\pi}^{\pi}\Phi\big(t,(r,0,z);s,(r'\cos\theta,r'\sin\theta,z')\big)\cdot r'\cos\theta\,d\theta.$$

Using the Gaussian upper bound (2.9) of the fundamental solution Φ, we get

Lemma 2.2 (Lemma 3.10 of [10]). *For any $\eta\in]0,1[$ and $0<s<t<T$, there exists some positive constant $C_{\eta,\alpha}$ depending only on the choice of η and $(\alpha_i)_{i=1}^n$, such that*

$$0<\widetilde{\Phi}(t,r,z;s,r',z')\le\frac{C_{\eta,\alpha}}{t-s}\Big|\frac{r'}{r}\Big|^{\frac12}\widetilde{H}\Big(\frac{t-s}{(1-\eta)rr'}\Big)e^{-\frac{1-\eta}{4(t-s)}\big((r-r')^2+(z-z')^2\big)}, \tag{2.12}$$

where $\widetilde{H}:(0,\infty)\to\mathbb{R}$ is decreasing, satisfies $\widetilde{H}(\tau)\le 1/\sqrt{\pi\tau}$ and $\widetilde{H}(\tau)\to1$ as $\tau\to0$ and $\widetilde{H}(\tau)\sim1/\sqrt{\pi\tau}$ as $\tau\to\infty$.

This Gaussian upper bound immediately transfers to ω^θ.

Proposition 2.3. *For any $\eta \in]0,1[$, $(r,z) \in \Omega$ and $0 < t < T$, we have*

$$0 < \omega_i^\theta(t,r,z) \leq \frac{C_{\eta,\alpha}}{t} e^{-\frac{1-\eta}{4t}\left((r-r_i)^2+(z-z_i)^2\right)}. \tag{2.13}$$

Proof. Using (2.12), we immediately get

$$0 < \omega_i^\theta(t,r,z) \leq \frac{C_{\eta,\alpha}}{t}\left|\frac{r_i}{r}\right|^{\frac{1}{2}} \widetilde{H}\left(\frac{t}{(1-\eta)rr_i}\right) e^{-\frac{1-\eta}{4t}\left((r-r_i)^2+(z-z_i)^2\right)}. \tag{2.14}$$

When $2r \leq r_i$, using the facts $\widetilde{H}(\tau) \leq 1/\sqrt{\pi\tau}$ and $2|r_i - r| \geq r_i$ in this case gives

$$\left|\frac{r_i}{r}\right|^{\frac{1}{2}} \widetilde{H}\left(\frac{t}{(1-\eta)rr_i}\right) \leq \frac{r_i}{\sqrt{\pi}}\left(\frac{1-\eta}{t}\right)^{\frac{1}{2}} \leq C_{\eta,\alpha}\cdot e^{\frac{\eta(1-\eta)}{4t}(r-r_i)^2}.$$

Substituting this into (2.14), and noting the fact that, when η runs over $]0,1[$, $(1-\eta)^2$ also runs over $]0,1[$, gives exactly (2.13) in this case.

And when $2r > r_i$, (2.13) follows by simply bounding $\widetilde{H}$ by 1 in (2.14). □

We would now like to take the limit $s \to 0$ in Eq. (2.11), in order to decompose the full solution ω^θ into the contributions $\omega_1^\theta, \ldots, \omega_n^\theta$ coming from the n vortex filaments. Keeping in mind the convergence $\omega^\theta(s,\cdot) \rightharpoonup \mu$ as $s \to 0$, we need at least the uniform continuity of $\widetilde{\Phi}$ in its last variable (r',z') up to the time boundary $s = 0$. This uniform continuity will stem from a uniform Hölder estimate on the original fundamental solution Φ, which we now state and prove.

Lemma 2.3. *The fundamental solution Φ is Hölder continuous in its last variable. More precisely, for every $\varepsilon > 0$ there exists a strictly positive α such that for any fixed x in $\mathbb{R}^3$ and any fixed t,s in $\mathbb{R}_+^*$ with $t - s > \varepsilon$, the function $y \mapsto \Phi(t,x;s,y)$ belongs to $\mathcal{C}^\alpha(\mathbb{R}^3)$. Moreover, the implied continuity constant depends solely on ε and $\|U\|_{L_t^\infty(L^\infty)_x^{-1}}$.*

Remark 2.1. The space $(L^\infty)^{-1}$ is defined as the space of all functions U which can be written as derivatives of L^∞ functions, i.e. for which there exist U_1, U_2 and U_3 such that

$$U = \partial_1 U^1 + \partial_2 U^2 + \partial_3 U^3.$$

A choice of norm on this space is given (for instance) by

$$\|U\|_{(L^\infty)^{-1}} = \inf_{(U_1,U_2,U_3)} \left(\|U_1\|_{L^\infty} + \|U_2\|_{L^\infty} + \|U_3\|_{L^\infty}\right),$$

the infimum being taken on all choices of (U_1,U_2,U_3) satisfying

$$U = \partial_1 U^1 + \partial_2 U^2 + \partial_3 U^3.$$

The space $L_t^\infty(L^\infty)_x^{-1}$ is then defined in an obvious way.

The idea of the proof is rather simple : we know that the corresponding result holds whenver the vector field V is identically zero. Hence, using a Grönwall argument, we can transfer the regularity statement to the general case. We begin by recalling some of the main results of H. Osada.

Theorem 2.1 (Theorems 1 and 2 in [12]). *Let $U : \mathbb{R}^3 \to \mathbb{R}^3$ be a divergence free vector field. If U belongs to $L_t^\infty(L^\infty)_x^{-1}$, then the equation*

$$\partial_t f + U \cdot \nabla f - \Delta f = 0$$

possesses a regular fundamental solution, Φ_0. This fundamental solutions satisfies the following properties.

(1) *For any $s < t$ and x, y in $\mathbb{R}^3$, there holds*

$$\int_{\mathbb{R}^3} \Phi_0(t,y;s,x)dx = \int_{\mathbb{R}^3} \Phi_0(t,y;s,x)dy = 1.$$

(2) *There exists constants* C_1, C_2, C_3, C_4 *such that for any* $s < t$ *and* x, y *in* $\mathbb{R}^3$*,*

$$C_1(t-s)^{-\frac{3}{2}} \exp\left(-C_2\frac{|x-y|^2}{t-s}\right) \le \Phi_0(t,x;s,y) \le C_3(t-s)^{-\frac{3}{2}} \exp\left(-C_4\frac{|x-y|^2}{t-s}\right).$$

(3) *There exists a strictly positive* α *such that for any* $\varepsilon > 0$*, there exists* $C > 0$ *depending only on* ε *and* $\|U\|_{L^\infty_t(L^\infty)^{-1}_x}$ *such that for every* x, x', y, y' *in* $\mathbb{R}^3$ *and every* t, t', s, s' *in* $\mathbb{R}^*_+$ *satisfying* $t-s, t'-s' > \varepsilon$*, there holds*

$$|\Phi_0(t,x;s,y) - \Phi_0(t',x';s',y')| \le C(|x-x'|^\alpha + |y-y'|^\alpha + |t-t'|^{\frac{\alpha}{2}} + |s-s'|^{\frac{\alpha}{2}}).$$

Of course, the results of H. Osada are stronger than the one we seek, though we will not need their full strength to achieve our goal.

We now state a Grönwall formula relating Φ and Φ_0.

Lemma 2.4. *For* $0 < s < t$ *and* x, y *in* $\mathbb{R}^3$*, The function* Φ *satisfies the equality*

$$\Phi(t,x;s,y) = \Phi_0(t,x;s,y) + \int_s^t \int_{\mathbb{R}^3} \Phi_0(t,x;s',z)\,[V(s',z)\Phi(s',z;s,y)]\,dzds'. \tag{2.15}$$

Proof. Let us denote by $\Psi(t,x;s,y)$ the right-hand side of the equality. As functions of (t,x) with (s,y) kept fixed, both Ψ ans Φ solve the affine equation

$$\partial_t f + U\cdot\nabla f - \Delta f = V\Phi.$$

Furthermore, they both satisfy

$$\Phi(t,\cdot;s,y), \Psi(t,\cdot;s,y) \rightharpoonup \delta_y \qquad \text{as } t \to s.$$

Since the vector fields U and V are smooth in space and time as long as $0 < s < t$, equality ensues. $\square$

We will conclude by appealing to the characterization of the Hölder space $\mathcal{C}^\alpha(\mathbb{R}^3)$ as the Besov space $B^\alpha_{\infty,\infty}$ and the (dyadic, inhomogeneous) Littlewood-Paley decomposition.

Proof of Lemma 2.3. For j in $\mathbb{N}$, let Δ_j be Littlewood-Paley projection around frequency 2^j. Applying Δ_j to each side of Eq. (2.15) in the last space variables gives

$$\Delta_j\Phi(t,x;s,y) = \Delta_j\Phi_0(t,x;s,y) + \int_s^t \int_{\mathbb{R}^3} \Phi_0(t,x;s',z)\,[V(s',z)(\Delta_j\Phi(s',z;s,y))]\,dzds'. \tag{2.16}$$

Taking absolute values and the supremum in y on each side, we get

$$\begin{aligned}\|\Delta_j\Phi(t,x;s,\cdot)\|_{L^\infty_y} &\le \|\Delta_j\Phi_0(t,x;s,\cdot)\|_{L^\infty_y}\\ &\quad + \int_s^t \int_{\mathbb{R}^3} \Phi_0(t,x;s',z)\left[\|V(s')\|_{L^\infty_x}\|\Delta_j\Phi(s',z;s,\cdot)\|_{L^\infty_y}\right] dzds'.\end{aligned} \tag{2.17}$$

Taking the supremum in x for the Φ_0 term outside the integral and the supremum in z for the Φ term inside the integral gives

$$\begin{aligned}\|\Delta_j\Phi(t,x;s,\cdot)\|_{L^\infty_y} &\le \|\Delta_j\Phi_0(t,\cdot;s,\cdot)\|_{L^\infty_{x,y}}\\ &\quad + \int_s^t \int_{\mathbb{R}^3} \Phi_0(t,x;s',z)\left[\|V(s')\|_{L^\infty_x}\|\Delta_j\Phi(s',\cdot;s,\cdot)\|_{L^\infty_{x,y}}\right] dzds'.\end{aligned} \tag{2.18}$$

Since Φ_0 has unit mass in the z variable, the above estimate simplifies itself into

$$\|\Delta_j\Phi(t,x;s,\cdot)\|_{L^\infty_y} \le \|\Delta_j\Phi_0(t,\cdot;s,\cdot)\|_{L^\infty_{x,y}} + \int_s^t \|V(s')\|_{L^\infty_x}\|\Delta_j\Phi(s',\cdot;s,\cdot)\|_{L^\infty_{x,y}} ds'. \tag{2.19}$$

The right-hand side does not depend anymore on x; hence, we may take the supremum in x in the left-hand side, leading to

$$\|\Delta_j\Phi(t,\cdot;s,\cdot)\|_{L^\infty_{x,y}} \le \|\Delta_j\Phi_0(t,\cdot;s,\cdot)\|_{L^\infty_{x,y}} + \int_s^t \|V(s')\|_{L^\infty_x}\|\Delta_j\Phi(s',\cdot;s,\cdot)\|_{L^\infty_{x,y}} ds'. \tag{2.20}$$

An immediate application of the Grönwall inequality yields

$$\|\Delta_j \Phi(t,\cdot;s,\cdot)\|_{L^\infty_{x,y}} \le \|\Delta_j \Phi_0(t,\cdot;s,\cdot)\|_{L^\infty_{x,y}} \exp\Big(\|V\|_{L^1_t L^\infty_x}\Big) \tag{2.21}$$

and since $\|V\|_{L^1_t L^\infty_x} < \infty$, the result follows. □

As an immediate consequence, the averaged fundamental solution $\widetilde{\Phi}$ is now unambiguously defined and Hölder continuous (with a possibly smaller exponent) at $s = 0$.

Let us write (2.11) in the following way

$$\begin{aligned}\omega^\theta(t,r,z) = &\int_\Omega \widetilde{\Phi}(t,r,z;0,r',z')\omega^\theta(s,r',z')\,dr'dz'\\ &+ \int_\Omega \big(\widetilde{\Phi}(t,r,z;s,r',z') - \widetilde{\Phi}(t,r,z;0,r',z')\big)\omega^\theta(s,r',z')\,dr'dz'.\end{aligned}$$

Combining the Hölder continuity of $\widetilde{\Phi}$ in (r',z') with the Gaussian upper bound on ω^θ and $\widetilde{\Phi}$, we know the second integral in the right-hand side converges to 0 as s tends to 0. On the other hand, since $(r',z') \mapsto \widetilde{\Phi}(t,r,z;0,r',z')$ is a continuous and bounded function for each (t,r,z), we may use Eq. (2.6). Taking the limit $s \to 0$ yields

$$\omega^\theta(t,r,z) = \int_\Omega \widetilde{\Phi}(t,r,z;0,r',z')\,d\mu(r',z').$$

Recalling $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$, the full vorticity ω^θ decomposes as follows.

$$\omega^\theta(t,r,z) = \sum_{i=1}^n \omega_i^\theta(t,r,z), \quad \text{where} \quad \omega_i^\theta(t,r,z) = \alpha_i \widetilde{\Phi}(t,r,z;0,r_i,z_i). \tag{2.22}$$

Correspondingly, the decomposition for $u = BS[\omega^\theta]$ writes

$$u(t,r,z) = \sum_{i=1}^n u_i(t,r,z), \quad \text{where} \quad u_i = BS[\omega_i^\theta]. \tag{2.23}$$

It is easy to see that $\omega_i^\theta \in C\big(]0,T[, L^1(\Omega)\cap L^\infty(\Omega)\big)$ is a mild solution of

$$\begin{cases} \partial_t \omega_i^\theta + u\cdot\nabla\omega_i^\theta - \big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\big)\omega_i^\theta = 0, & (t,r,z) \in]0,T[\times\Omega,\\ \omega_i^\theta \rightharpoonup \alpha_i\delta_{x_i} \quad \text{as} \quad t\to 0. \end{cases} \tag{2.24}$$

Moreover, we have the following estimates for ω_i^θ.

Proposition 2.4.
(i)
$$\|\omega_i^\theta(t)\|_{L^1(\Omega)} \le \|\mu\|_{\mathrm{tv}} \quad \textit{and} \quad \lim_{t\to 0}\|\omega_i^\theta(t)\|_{L^1(\Omega)} = \alpha_i. \tag{2.25}$$

(ii) *There exists some positive time $t_1 < T$, such that for any $0 < t < t_1$, there holds*

$$t^{\frac{3}{2}}\|\nabla\omega_i^\theta(t)\|_{L^\infty(\Omega)} \le C_0. \tag{2.26}$$

Proof. (i) To prove (2.25), notice that $\omega_i^\theta > 0$ and $\omega^\theta = \sum_{i=1}^n \omega_i^\theta$, we have

$$\sum_{i=1}^n \|\omega_i^\theta(t)\|_{L^1(\Omega)} = \|\omega^\theta(t)\|_{L^1(\Omega)} \le \|\mu\|_{\mathrm{tv}}, \quad \forall t \in]0,T[, \tag{2.27}$$

which in particular implies $\|\omega_i^\theta(t)\|_{L^1(\Omega)} \le \|\mu\|_{\mathrm{tv}}$. By taking limit $t \to 0$ in (2.27), we obtain

$$\sum_{i=1}^n \lim_{t\to 0}\|\omega_i^\theta(t)\|_{L^1(\Omega)} = \lim_{t\to 0}\|\omega^\theta(t)\|_{L^1(\Omega)} = \|\mu\|_{\mathrm{tv}} = \sum_{i=1}^n \alpha_i.$$

On the other hand, the initial condition $\omega_i^\theta \rightharpoonup \alpha_i \delta_{x_i}$ as $t \to 0$ implies

$$\lim_{t\to 0} \|\omega_i^\theta(t)\|_{L^1(\Omega)} \geq \alpha_i.$$

Combining the above two sides, clearly there must hold

$$\lim_{t\to 0} \|\omega_i^\theta(t)\|_{L^1(\Omega)} = \alpha_i.$$

(ii) For any $0 < t < T$, we first write (2.24) in the integral form as

$$\omega_i^\theta(t) = S(t/2)\omega_i^\theta(t/2) - \int_{t/2}^t S(t-s)\,\mathrm{div}_*\big(u(s)\omega_i^\theta(s)\big)\,ds. \tag{2.28}$$

Then we need the following lemma.

Lemma 2.5. *For any $1 \leq p \leq q \leq \infty$, and $f(r,z) \in L^p(\Omega)$, there holds*

$$\|\nabla S(t) f\|_{L^q(\Omega)} \leq \frac{C}{t^{\frac12+\frac1p-\frac1q}} \|f\|_{L^p(\Omega)}, \tag{2.29}$$

Proof. By using (2.2), let us first write the explicit formula of $\nabla S(t) f$ as

$$\begin{aligned}(\partial_r,\ \partial_z)\,(S(t)f) = \frac{1}{4\pi t}\int_\Omega \frac{\bar r^{\frac12}}{r^{\frac12}} \exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}\right)\cdot f(\bar r,\bar z)\\ \cdot\left(-\frac{t}{r^2\bar r}H'\left(\frac{t}{r\bar r}\right) - \left(\frac{1}{2r}+\frac{r-\bar r}{2t}\right)H\left(\frac{t}{r\bar r}\right),\right.\\ \left.-\frac{z-\bar z}{2t}H\left(\frac{t}{r\bar r}\right)\right)\,d\bar r\,d\bar z.\end{aligned} \tag{2.30}$$

Let us denote

$$B_1(r,z,\bar r,\bar z) \stackrel{\text{def}}{=} \left(\frac{t}{r^{\frac52}\bar r^{\frac12}}\left|H'\left(\frac{t}{r\bar r}\right)\right| + \frac{\bar r^{\frac12}}{r^{\frac32}}\left|H\left(\frac{t}{r\bar r}\right)\right|\right)\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}\right),$$

$$B_2(r,z,\bar r,\bar z) \stackrel{\text{def}}{=} \frac{\bar r^{\frac12}}{r^{\frac12}}\left(\left|\frac{r-\bar r}{2t}H\left(\frac{t}{r\bar r}\right)\right| + \left|\frac{z-\bar z}{2t}H\left(\frac{t}{r\bar r}\right)\right|\right)\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}\right).$$

When $\bar r \leq 2r$, we deduce from the bound for H and H' that

$$\begin{aligned}B_1 &\lesssim \left(\frac{t}{r^{\frac52}\bar r^{\frac12}}\left|\frac{r\bar r}{t}\right|^{\frac32} + \frac{\bar r^{\frac12}}{r^{\frac32}}\left|\frac{r\bar r}{t}\right|^{\frac12}\right)\cdot\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}\right)\\ &\lesssim \frac{1}{t^{\frac12}}\cdot\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}\right),\end{aligned}$$

and

$$\begin{aligned}B_2 &\lesssim \left(\left|\frac{r-\bar r}{2t}\right| + \left|\frac{z-\bar z}{2t}\right|\right)\cdot\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}\right)\\ &\lesssim \frac{1}{t^{\frac12}}\cdot\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{5t}\right),\end{aligned}$$

While for the case $\bar r > 2r$, there holds $\bar r < 2|\bar r - r|$, thus we can get

$$\begin{aligned}B_1 &\lesssim \left(\frac{t}{r^{\frac52}\bar r^{\frac12}}\left|\frac{r\bar r}{t}\right|^{\frac52} + \frac{\bar r^{\frac12}}{r^{\frac32}}\left|\frac{r\bar r}{t}\right|^{\frac32}\right)\cdot\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}\right)\\ &\lesssim \frac{\bar r^2}{t^{\frac32}}\cdot\left(\frac{t}{(r-\bar r)^2+(z-\bar z)^2}\right)\cdot\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{5t}\right)\\ &\lesssim \frac{1}{t^{\frac12}}\cdot\exp\left(-\frac{(r-\bar r)^2+(z-\bar z)^2}{5t}\right),\end{aligned}$$

and

$$
\begin{aligned}
B_2 &\lesssim \frac{\bar{r}^{\frac{1}{2}}}{r^{\frac{1}{2}}} \left(\left| \frac{r-\bar{r}}{2t} \right| \left| \frac{r\bar{r}}{t} \right|^{\frac{1}{2}} + \left| \frac{z-\bar{z}}{2t} \right| \left| \frac{r\bar{r}}{t} \right|^{\frac{1}{2}} \right) \cdot \exp\left(-\frac{(r-\bar{r})^2+(z-\bar{z})^2}{4t} \right) \\
&\lesssim \frac{1}{t^{\frac{1}{2}}} \cdot \exp\left(-\frac{(r-\bar{r})^2+(z-\bar{z})^2}{5t} \right),
\end{aligned}
$$

Combining the above estimates, we always have

$$
B_1(r,z,\bar{r},\bar{z}) + B_2(r,z,\bar{r},\bar{z}) \lesssim \frac{1}{t^{\frac{1}{2}}} \cdot \exp\left(-\frac{(r-\bar{r})^2+(z-\bar{z})^2}{5t} \right). \tag{2.31}
$$

Then (2.29) comes from (2.30), (2.31), and Young's inequality. □

Using (2.28) and (2.29), together with the bounds (2.4) and (2.5), as well as the pointwise bound $\omega_i^\theta \le \omega^\theta$, we achieve

$$
\begin{aligned}
\|\nabla\omega_i^\theta(t)\|_{L^\infty(\Omega)} &\le \frac{C}{t^{3/2}} \|\omega_i^\theta(t/2)\|_{L^1(\Omega)} + \int_{\frac{t}{2}}^t \frac{C}{(t-s)^{1/2}} \big(\|\nabla u(s)\|_{L^\infty(\Omega)} \|\omega_i^\theta(s)\|_{L^\infty(\Omega)} \\
&\quad + \|u(s)\|_{L^\infty(\Omega)} \|\nabla\omega_i^\theta(s)\|_{L^\infty(\Omega)} \big)\, ds \\
&\le \frac{C_0}{t^{3/2}} \|\mu\|_{tv} + \int_{\frac{t}{2}}^t \frac{C_0}{(t-s)^{1/2}} \left(\frac{1}{s^2} + \frac{1}{\sqrt{s}} \cdot s^{\frac{1}{2}} \|u(s)\|_{L^\infty(\Omega)} \|\nabla\omega_i^\theta(s)\|_{L^\infty(\Omega)} \right) ds \\
&\le \frac{C_0}{t^{3/2}} (1+\|\mu\|_{tv}) + \int_{\frac{t}{2}}^t \frac{1}{s^2} \frac{C_0}{(t-s)^{1/2}} \left(s^{\frac{1}{2}} \|u(s)\|_{L^\infty(\Omega)} s^{\frac{3}{2}} \|\nabla\omega_i^\theta(s)\|_{L^\infty(\Omega)} \right) ds.
\end{aligned}
$$

Multiplying both sides by $t^{3/2}$, we get

$$
t^{\frac{3}{2}} \|\nabla\omega_i^\theta(t)\|_{L^\infty(\Omega)} \le C_0(1+\|\mu\|_{tv}) + \int_{\frac{t}{2}}^t \frac{t^{\frac{3}{2}}}{s^{\frac{3}{2}}} \frac{C_0}{\sqrt{s(t-s)}} \left(s^{\frac{1}{2}} \|u(s)\|_{L^\infty(\Omega)} s^{\frac{3}{2}} \|\nabla\omega_i^\theta(s)\|_{L^\infty(\Omega)} \right) ds.
$$

Hence, an applicaton of Grönwall's inequality to the function $t \mapsto t^{\frac{3}{2}} \|\nabla\omega_i^\theta(t)\|_{L^\infty(\Omega)}$ yields

$$
t^{\frac{3}{2}} \|\nabla\omega_i^\theta(t)\|_{L^\infty(\Omega)} \le C_0(1+\|\mu\|_{tv}) \exp\left(\int_{\frac{t}{2}}^t \frac{t^{\frac{3}{2}}}{s^{\frac{3}{2}}} \frac{C_0}{\sqrt{s(t-s)}} \left(s^{\frac{1}{2}} \|u(s)\|_{L^\infty(\Omega)} \right) ds \right).
$$

Owing to the bound (2.4), we finally have

$$
t^{\frac{3}{2}} \|\nabla\omega_i^\theta(t)\|_{L^\infty(\Omega)} \le C_0(1+\|\mu\|_{tv}) \exp\left(\int_{\frac{t}{2}}^t \frac{t^{\frac{3}{2}}}{s^{\frac{3}{2}}} \frac{C_0}{\sqrt{s(t-s)}}\, ds \right) < \infty.
$$

This gives exactly the desired estimate (2.26). □

2.2. Self-similar Variables

In view of (2.13), we know that ω_j^θ concentrates in a self-similar way around x_j for short time. Thus it is very natural to introduce the self-similar variables:

$$
R_j = \frac{r-r_j}{\sqrt{t}}, \quad Z_j = \frac{z-z_j}{\sqrt{t}}, \quad X_j = \frac{x-x_j}{\sqrt{t}} \quad \text{and} \quad \epsilon_j = \frac{\sqrt{t}}{r_j}, \quad j = 1, \ldots, n. \tag{2.32}
$$

Correspondingly, for any $j \in \{1, \ldots, n\},\ t \in (0,T)$ and any $(r,z) \in \Omega$, we set

$$
\omega_j^\theta(t,r,z) = \frac{\alpha_j}{t} f_j\left(t, \frac{r-r_j}{\sqrt{t}}, \frac{z-z_j}{\sqrt{t}} \right), \quad u_j(t,r,z) = \frac{\alpha_j}{\sqrt{t}} U_j\left(t, \frac{r-r_j}{\sqrt{t}}, \frac{z-z_j}{\sqrt{t}} \right). \tag{2.33}
$$

In the new coordinates (R_j, Z_j), the domain constraint $r > 0$ translates into $r_j + \sqrt{t}R_j > 0$, which means that the rescaled vorticity $f_j(t, R_j, Z_j)$ is defined in the time-dependent domain

$$\Omega_{\epsilon_j} \stackrel{\text{def}}{=} \left\{(R_j, Z_j) \in \mathbb{R}^2 \mid 1 + \epsilon_j R_j > 0\right\}.$$

Noting that $u_j = BS[\omega_j^\theta]$, thus U_j can also be determined by f_j. Recalling the subsection 4.2 of [10], we have the following explicit representation

$$\begin{aligned} U_j^r(X_j) &= \frac{1}{2\pi}\int_{\Omega_{\epsilon_j}} \sqrt{(1+\epsilon_j R')(1+\epsilon_j R_j)^{-1}} F_1(\xi_j^2)\frac{Z_j - Z'}{|X_j - X'|^2} f_j(X')\,dX', \\ U_j^z(X_j) &= -\frac{1}{2\pi}\int_{\Omega_{\epsilon_j}} \sqrt{(1+\epsilon_j R')(1+\epsilon_j R_j)^{-1}} F_1(\xi_j^2)\frac{R_j - R'}{|X_j - X'|^2} f_j(X')\,dX' \\ &\quad + \frac{\epsilon_j}{4\pi}\int_{\Omega_{\epsilon_j}} \sqrt{(1+\epsilon_j R')(1+\epsilon_j R_j)^{-3}} \big(F_1(\xi_j^2) + F_2(\xi_j^2)\big) f_j(X')\,dX', \end{aligned} \tag{2.34}$$

where $F_1,\ F_2$ is some kernel satisfying $s^{\sigma_1}F_1(s),\ s^{\sigma_2}F_2(s)$ are bounded on $]0,\infty[$ whenever $0 \le \sigma_1 \le 3/2,\ 0 < \sigma_2 \le 3/2$, and ξ_j^2 is a shorthand notation for the quantity

$$\xi_j^2 = \epsilon_j^2 |X_j - X'|^2 (1+\epsilon_j R_j)^{-1}(1+\epsilon_j R')^{-1}.$$

We denote this map from f_j to U_j by $U_j = BS^{\epsilon_j}[f_j]$. We use the superscript ϵ_j since in the new variables, the map depends explicitly on time through the parameter ϵ_j.

In the rest of this paper, the following notations will also be used:

$$R = \frac{r - r_i}{\sqrt{t}}, \quad Z = \frac{z - z_i}{\sqrt{t}}, \quad X = \frac{x - x_i}{\sqrt{t}} \quad \text{and} \quad \epsilon = \frac{\sqrt{t}}{r_i}, \tag{2.35}$$

here although $R,\ Z,\ X,\ \epsilon$ indeed depend on i, we omit the index i for notation simplification.

After this blow-up procedure, the gaussian bound on ω_i given by (2.13) translates into

$$0 < f_i(t, R, Z) \le C_{\eta,\alpha} e^{-\frac{1-\eta}{4}(R^2+Z^2)}, \tag{2.36}$$

and (2.25) translates into

$$\int_{\Omega_\epsilon} f_i(t, R, Z)\,dRdZ \to 1, \quad \text{as} \quad t \to 0. \tag{2.37}$$

We can use the estimate (2.36) to derive the pointwise estimate for U_i^ϵ. First, recalling the proof of Proposition 2.3 in [9], which shows that for any $(r, z) \in \Omega$, there holds

$$|u(r, z)| \le C\int_\Omega \frac{1}{\sqrt{(r-r')^2 + (z-z')^2}} |\omega^\theta(r', z')|\,dr'dz'.$$

Then using the self-similar variables (2.32), we obtain

$$|U_i(t, R, Z)| \le C\int_{\Omega_{\epsilon_i}} \frac{1}{\sqrt{(R-R')^2 + (Z-Z')^2}} f_i(t, R', Z')\,dR'dZ'$$

Finally substituting (2.36) with some fixed η into this, leads to

$$\big(1 + |R| + |Z|\big)|U_i(t, R, Z)| \le C_0. \tag{2.38}$$

Using the notation (2.33), let us also do this self-similar blow-up of the whole velocity u near the point $x_i \in \Omega$ and near the initial time $t = 0$, and we get

$$u(t, r, z) = \frac{\alpha_i}{\sqrt{t}} U_i(t, R, Z) + \sum_{j \ne i} \frac{\alpha_j}{\sqrt{t}} U_j\left(t, R + \frac{r_i - r_j}{\sqrt{t}}, Z + \frac{z_i - z_j}{\sqrt{t}}\right). \tag{2.39}$$

In view of (2.38), let $t \to 0$ and $R,\ Z$ fixed, all $U_j\big(t, R + \frac{r_i - r_j}{\sqrt{t}}, Z + \frac{z_i - z_j}{\sqrt{t}}\big)$ for $j \ne i$ vanish, since they are localized far away from the center of the blow-up procedure and decay exponentially fast, in particular much faster than the multiplicative term $\frac{\alpha_i}{\sqrt{t}}$ and only $U_i(t, R, Z)$ remains. Thus after this blow-up procedure, the convection term can be very close to $U_i \cdot \nabla f_i$, for a short time. Combining with the fact that the initial measure for $\omega_i = \omega_i^\theta e_\theta$ is $\alpha_i \delta_{x_i}$, hence if we believe in uniqueness, it is reasonable to expect that, for a short time, ω_i will be very close to an Oseen vortex located at x_i with circulation α_i.

In order to write this observation precisely, let us denote the following functions on $\mathbb{R}^2$:

$$w(x,y) \stackrel{\text{def}}{=} e^{(|x|^2+|y|^2)/4}, \quad G(x,y) \stackrel{\text{def}}{=} \frac{1}{4\pi} e^{-(|x|^2+|y|^2)/4}, \quad (x,y) \in \mathbb{R}^2,$$

and denote by $\mathcal{X}$ the weighted space $L^2(\mathbb{R}^2, w(x,y)dxdy)$. We have:

Proposition 2.1. *For any $i \in \{1,\ldots,n\}$, we have $\|\overline{f}_i(t,\cdot) - G(\cdot)\|_{\mathcal{X}} \to 0$ as t goes to 0, where $\overline{f}_i$ denotes the extension of f_i by zero outside Ω_ϵ.*

Proof. First, let us denote by $\mathcal{X}_0$ a subspace of $\mathcal{X}$, which is defined by the stronger norm

$$\|f\|_{\mathcal{X}_0} \stackrel{\text{def}}{=} \|fw^{1-\eta}\|_{L^\infty(\mathbb{R}^2)} + \|\nabla f\|_{L^\infty(\mathbb{R}^2)},$$

where η is a real number satisfying $0 < \eta < \frac{1}{2}$. We have:

Lemma 2.1 (Lemma 4.4 in [10]). *The space $\mathcal{X}_0$ is compactly embedded in $\mathcal{X}$, and the unit ball in $\mathcal{X}_0$ is closed for the topology induced by $\mathcal{X}$.*

In the self-similar variables, the gradient bound for ω_i^θ, namely (2.26), translates into

$$\|\nabla \overline{f}_i(t)\|_{L^\infty(\mathbb{R}^2)} < \infty, \quad \forall t \in]0,T[.$$

Combining this with the gaussian bound for f_i, (2.36), we know that, $(\overline{f}_i(t))_{0<t<T}$ is a bounded subset of $\mathcal{X}_0$, hence compact in $\mathcal{X}$. Let h_* be an accumulation point in $\mathcal{X}$ of $(\overline{f}_i(t))_{0<t<T}$ as t goes to 0, and $(t_m)_{m\in\mathbb{N}}$ be the corresponding sequence of positive time satisfying

$$t_m \to 0, \quad \|\overline{f}_i(t_m) - h_*\|_{\mathcal{X}} \to 0 \quad \text{as} \quad m \to \infty. \tag{2.40}$$

Now, let us temporarily consider the whole 3-D vorticity field ω and the whole 3-D velocity field u. For any $m \in \mathbb{N}$, $y \in \mathbb{R}^3$, and $s \in]0, t_m^{-1}T[$, we define the following sequence

$$\begin{cases} u^{(m)}(s,y) = \sqrt{t_m}\, u(t_m s, x_i + \sqrt{t_m} y) \\ \omega^{(m)}(s,y) = t_m \omega(t_m s, x_i + \sqrt{t_m} y), \end{cases}$$

where $x_i = (r_i, 0, z_i) \in \mathbb{R}^3$. In other words, the vector fields $\omega^{(m)}$, $u^{(m)}$ are defined by a self-similar blow-up of the original quantities ω, u near the point $x_i \in \mathbb{R}^3$ and near the initial time $t = 0$. It is easy to verify that ω, u satisfy the 3-D vorticity equation:

$$\partial_s \omega^{(m)} + u^{(m)} \cdot \nabla \omega^{(m)} - \Delta \omega^{(m)} = \omega^{(m)} \cdot \nabla u^{(m)}, \quad \operatorname{div} u^{(m)} = 0, \quad \operatorname{curl} u^{(m)} = \omega^{(m)},$$

for $s \in]0, t_m^{-1}T[$, $y \in \mathbb{R}^3$. The self-similar rescaling from u to $u^{(m)}$ preserves the bounds given by (2.4), precisely for all indices k, $\ell \in \mathbb{N}$, we have the following *a priori* estimates

$$\|\partial_s^k \nabla_y^\ell u^{(m)}(s)\|_{L^\infty(\mathbb{R}^3)} \le C_0 s^{-\left(\frac{1}{2}+k+\frac{\ell}{2}\right)}, \quad s \in]0, t_m^{-1}T[,$$

which holds uniformly in m. Hence, up to an extraction, we can assume that

$$\omega^{(m)} \to \overline{\omega}, \quad u^{(m)} \to \overline{u}, \quad \text{as} \quad m \to \infty,$$

with uniform convergence of both vector fields along with all their derivatives on any compact subset of $]0, t_m^{-1}T[\times \mathbb{R}^3$. Thus the limiting fields $\overline{\omega}$, $\overline{u}$ are smooth and satisfy

$$\partial_s \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} - \Delta \overline{\omega} = \overline{\omega} \cdot \nabla \overline{u}, \quad \operatorname{div} \overline{u} = 0, \quad \operatorname{curl} \overline{u} = \overline{\omega}. \tag{2.41}$$

The goal now is to relate $\overline{\omega}$ to ω_i and $\overline{f}_i$. The idea is that the other ω_j, $\overline{f}_j$ $(j \neq i)$ should be eliminated by the blow-up procedure. Using the definitions, we get

$$\begin{aligned}\omega^{(m)}(s,y) &= t_m\omega(t_m s, x_i + \sqrt{t_m}y)\\ &= t_m\omega(t_m s, \sqrt{(r_i+\sqrt{t_m}y_1)^2 + t_m y_2^2}, 0, z_i + \sqrt{t_m}y_3)\\ &= \left(\frac{\alpha_i}{s}\overline{f}_i(t_m s, X_{ii}^{(m)}(s,y)) + \sum_{j\neq i}\frac{\alpha_j}{s}\overline{f}_j\left(t_m s, X_{ij}^{(m)}(s,y)\right)\right)e_\theta(x_i+\sqrt{t_m}y),\end{aligned} \tag{2.42}$$

where

$$X_{ij}^{(m)}(s,y) \overset{\text{def}}{=} \left(\frac{\sqrt{(r_i+\sqrt{t_m}y_1)^2 + t_m y_2^2} - r_j}{\sqrt{t_m s}}, \frac{z_i - z_j + \sqrt{t_m}y_3}{\sqrt{t_m s}}\right).$$

If $i \neq j$, for any bounded subset $B \subset \mathbb{R}^3$ and any $y \in B$, there exists a large constant N_B, such that for any $m > N_B$, there holds

$$|X_{ij}^{(m)}(s,y)|^2 \geq \frac{(r_i-r_j)^2 + (z_i-z_j)^2}{2t_m s}.$$

Then the gaussian bound for f_j (2.36) entails

$$0 \leq \overline{f}_j\left(t_m s, X_{ij}^{(m)}(s,y)\right) \leq C_{\eta,\alpha}\exp\Big\{-\frac{(1-\eta)|x_i-x_j|^2}{8t_m s}\Big\}.$$

Hence, the only contribution in the limit procedure $m \to \infty$ comes, as expected, from the i-th circular vortex. Regarding $\overline{f}_i$, as shown before, $\overline{f}_i(\cdot,\cdot,t)$ is bounded in $\mathcal{X}_0$. Thus for any fixed $s > 0$, up to another extraction, there must exist some $h_s \in \mathcal{X}$ such that

$$\|\overline{f}_i(t_m s) - h_s\|_{\mathcal{X}} \to 0 \quad \text{as} \quad m \to \infty. \tag{2.43}$$

The boundedness of $(\overline{f}_i(t_m s))_m$ in $\mathcal{X}_0$ implies that, this convergence of $(\overline{f}_i(t_m s))_m$ to h_s also holds uniformly on any compact set of $\mathbb{R}^3$. Therefore, taking the limit $m \to \infty$ on both sides of (2.42) and noting that $e_\theta(x_i) = e_2 = (0,1,0)$, we obtain

$$\overline{\omega}(s,y) = \frac{\alpha_i}{s}h_s\left(\frac{y_1}{\sqrt{s}}, \frac{y_3}{\sqrt{s}}\right)e_2 \overset{\text{def}}{=} (0, \overline{\omega}_2(s,y_1,y_3), 0).$$

Taking the limit $m \to \infty$ in (2.36) and (2.37), we deduce

$$|\overline{\omega}_2(s,y_1,y_3)| \lesssim C_{\eta,\alpha}s^{-1}e^{-\frac{1-\eta}{4s}|y|^2}, \qquad \int_{\mathbb{R}^2}\overline{\omega}_2(s,y_1,y_3)\,dy_1dy_3 = \alpha_i. \tag{2.44}$$

We now turn to the velocity field. Similarly as (2.42), we can write

$$u^{(m)}(s,y) = \frac{\alpha_i}{\sqrt{s}}U_i^\epsilon\left(t_m s, X_{ii}^{(m)}(s,y)\right) + \sum_{j\neq i}\frac{\alpha_j}{\sqrt{s}}U_j^\epsilon(t_m s, X_{ij}^{(m)}(s,y)). \tag{2.45}$$

In view of (2.38), as $t_m \to 0$, all $U_j^\epsilon(t_m s, X_{ij}^{(m)}(s,y))$ for $j \neq i$ vanish, and only $U_i^\epsilon(t_m s, X_{ii}^{(m)}(s,y))$ remains. Regarding U_i, using (2.38) again and taking the limit $m \to \infty$, we get

$$|\overline{u}(s,y)| \lesssim (\sqrt{s} + |y_1| + |y_3|)^{-1}. \tag{2.46}$$

Moreover, as shown in (2.41), $\overline{u}$ satisfies the following elliptic system

$$\operatorname{div}\overline{u} = 0, \quad \operatorname{curl}\overline{u} = \overline{\omega}.$$

This div-curl system has at most one solution with the decay property (2.46), hence

$$\overline{u}(s,y) = \overline{u}_1(s,y_1,y_3)e_1 + \overline{u}_3(s,y_1,y_3)e_3 = (\overline{u}_1(s,y_1,y_3), 0, \overline{u}_3(s,y_1,y_3)),$$

where $(\overline{u}_1, \overline{u}_3)$ is the two dimensional velocity field obtained from the scalar vorticity $\overline{\omega}_2$ via the Biot-Savart law in $\mathbb{R}^2$.

Summarizing, we have shown that the limiting vorticity $\overline{\omega}_2$, together with the associated velocity $(\overline{u}_1, \overline{u}_3)$ solves the 2-D Navier–Stokes equations, and it follows from (2.44) that $\overline{\omega}_2(s,\cdot)$ is uniformly bounded in $L^1(\mathbb{R}^2)$ and converges weakly to the Dirac measure $\alpha_i\delta_0$ as $s \to 0$. Then we deduce, by using Proposition 1.3 in [11], that $\overline{\omega}_2(s, y_1, y_3) = \frac{\alpha_i}{s} G\left(\frac{y_1}{\sqrt{s}}, \frac{y_3}{\sqrt{s}}\right)$, i.e. $h_s = G$ for any $s > 0$. In particular, choosing $s = 1$ so that $t_m s = t_m$, and comparing (2.40) with (2.43), we conclude that $h_* = G$, which is the desired result. □

In view of Proposition 2.1, it is natural to make a further decomposition of ω. Let

$$d \stackrel{\text{def}}{=} \min_{1\le i<j\le n} \big\{|x_i - x_j|,\, r_i\big\},$$

and $\chi : [0,\infty[\to [0,1]$ to be a smooth non-increasing cutoff function such that $\chi = 1$ on $[0,1/8]$ and χ vanishes outside $[0,1/4]$. Let f_0 to be a function on $]0,T[\times \mathbb{R}^2$ defined as

$$f_0(t,x,y) \stackrel{\text{def}}{=} G(x,y)\chi\big(\sqrt{t(x^2+y^2)}/d\big), \quad (x,y) \in \mathbb{R}^2,\ t \in]0,T[,$$

and $\widetilde{f}_i$ to be a function on $]0,T[\times \Omega^i_\epsilon$ defined as

$$\widetilde{f}_i(t,R,Z) = f_i(t,R,Z) - f_0(t,R,Z), \quad (R,Z) \in \Omega_\epsilon,\ t \in]0,T[. \tag{2.47}$$

Then we can decompose ω^θ further as follows:

$$\omega^\theta(t,r,z) = \sum_{j=1}^n \left(\frac{\alpha_j}{t} f_0(t,R_j,Z_j) + \frac{\alpha_j}{t}\widetilde{f}_j(t,R_j,Z_j)\right). \tag{2.48}$$

And correspondingly, $u = BS[\omega^\theta]$ can be decomposed further into

$$\begin{gathered} u(t,r,z) = \sum_{j=1}^n \left(\frac{\alpha_j}{\sqrt{t}} U_{0,j}(t,R_j,Z_j) + \frac{\alpha_j}{\sqrt{t}}\widetilde{U}_j(t,R_j,Z_j)\right), \quad \text{where} \\ U_{0,j} = BS^{\epsilon_j}[f_0], \quad \widetilde{U}_j = BS^{\epsilon_j}[\widetilde{f}_j]. \end{gathered} \tag{2.49}$$

Remark 2.2. For any $j \in \{1,\dots,n\}$, due to the cutoff function χ, it is easy to see that $f_0(t,R_j,Z_j)$ vanishes when $\sqrt{t}R_j < -d/4$, and thus vanishes when $\sqrt{t}R_j < -r_j/4$. In particular, this implies that $f_0(t,R_j,Z_j)$ satisfies the Dirichlet boundary condition on $\partial\Omega_{\epsilon_j}$, and thus $\widetilde{f}_j(t,R_j,Z_j)$ also satisfies the Dirichlet boundary condition on $\partial\Omega_{\epsilon_j}$.

It is clear that $f_0(t) \in \mathcal{X}$ for all $t \in]0,T[$, and $\|f_0(t) - G\|_{\mathcal{X}} \to 0$ as $t \to 0$. Thus the perturbation $\widetilde{f}_j(t)$ (extended by zero outside Ω_{ϵ_j}) belongs to $\mathcal{X}$ for all $t \in]0,T[$, and Proposition 2.1 implies that $\|\widetilde{f}_j(t)\|_{\mathcal{X}} \to 0$ as $t \to 0$. We recall that the weighted L^2 norm of $\mathcal{X}$ controls the standard L^1 norm, by a straightfroward application of the Hölder inequality. In the next section, we shall quantify this rate of convergence.

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In view of the decomposition (2.48), to prove the uniqueness claim in Theorem 1.1, we only need to show the perturbation part $(\widetilde{f}_j)_{j=1}^n$ is uniquely determined. At the end of last section, we have shown that $\|\widetilde{f}_j(t)\|_{\mathcal{X}} \to 0$ as $t \to 0$, but this is not enough to prove uniqueness. We shall give a more accurate quantitative rate of this convergence, which in particular implies the short time estimate (1.5). This will be done in the first subsection.

After some modifications to the energy estimates in the proof of the short time estimate, we can prove the uniqueness claim in Theorem 1.1. This will be done in the second subsection.

3.1. Short Time Asymptotics

Using (2.24) and (2.33), we can derive the evolution equation satisfied by the rescaled vorticity f_i reads

$$t\partial_t f_i(t,X) + \mathrm{div}_*\big(\alpha_i U_i(t,X) f_i(t,X) + W_i(t,X) f_i(t,X)\big) = (\mathcal{L} f_i)(t,X) + \partial_R\Big(\frac{\epsilon f_i(t,X)}{1+\epsilon R}\Big), \tag{3.1}$$

for $X \in \Omega_\epsilon$ and $t \in]0,T[$, where the operator $\mathcal{L}$ is defined for a generic function f by

$$\mathcal{L} f(X) \stackrel{\text{def}}{=} \Delta_X f(X) + \frac{X}{2}\cdot\nabla_X f(X) + f(X),$$

the operator div_* is defined for a generic vector field $V(X) = V^r(X)e_r + V^z(X)e_z$ by

$$\mathrm{div}_*\big(V(X)\big) \stackrel{\text{def}}{=} \partial_R V^r(X) + \partial_Z V^z(X),$$

and W_i stands for the other parts of the rescaled velocity:

$$W_i(t,X) \stackrel{\text{def}}{=} \sum_{j\neq i} \alpha_j U_j(t,X_j), \quad \text{where} \quad X_j = \frac{x-x_j}{\sqrt{t}} = X + \frac{x_i - x_j}{\sqrt{t}}.$$

Then we can deduce from (2.48), (2.49) and (3.1) that

$$t\partial_t \widetilde{f}_i + \alpha_i\,\mathrm{div}_*(U_{0,i}\widetilde{f}_i + \widetilde{U}_i f_0 + \widetilde{U}_i\widetilde{f}_i) + \mathrm{div}_*(W_i f_i) = \mathcal{L}\widetilde{f}_i + \partial_R\left(\frac{\epsilon\widetilde{f}_i}{1+\epsilon R}\right) + \mathcal{H}, \tag{3.2}$$

where

$$\mathcal{H} = -t\partial_t f_0 + \mathcal{L} f_0 + \partial_R\Big(\frac{\epsilon f_0(t,X)}{1+\epsilon R}\Big) - \alpha_i\,\mathrm{div}_*(U_{0,i} f_0).$$

And we shall define, following [10], the two types of energy for each vortex

$$\begin{aligned} E_j(t) &\stackrel{\text{def}}{=} \frac{1}{2}\int_{\Omega_{\epsilon_j}} \widetilde{f}_j(t,X_j)^2 w(X_j)\,dX_j, \\ \mathcal{E}_j(t) &\stackrel{\text{def}}{=} \frac{1}{2}\int_{\Omega_{\epsilon_j}} \Big(|\nabla\widetilde{f}_j(t,X_j)|^2 + (1+|X_j|^2)\widetilde{f}_j(t,X_j)^2\Big) w(X_j)\,dX_j, \end{aligned} \tag{3.3}$$

as well as the total energies

$$E(t) \stackrel{\text{def}}{=} \sum_{j=1}^n E_j(t), \qquad \mathcal{E}(t) \stackrel{\text{def}}{=} \sum_{j=1}^n \mathcal{E}_j(t).$$

As we have pointed out in Remark 2.2 that, $\widetilde{f}_j$ satisfies the homogeneous Dirichlet condition on $\partial\Omega_{\epsilon_j}$, thus although the integral in (3.3) is taken over the time-dependent domain Ω_{ϵ_j}, there is no contribution from the boundary when we differentiate with respect to time. Hence we can get, by doing $L^2(\Omega_\epsilon, w(X)dX)$ energy estimate to (3.2) and integrating by parts, that

$$tE_i'(t) = A_i(t) + I_i(t), \tag{3.4}$$

where

$$\begin{aligned} A_i(t) = \int_{\Omega_\epsilon} &\left(\mathcal{L}\widetilde{f}_i(t,X) + \partial_R\left(\frac{\epsilon\widetilde{f}_i(t,X)}{1+\epsilon R}\right) + \mathcal{H}(t,X)\right. \\ &\left. - \alpha_i\,\mathrm{div}_*(U_{0,i}\widetilde{f}_i + \widetilde{U}_i f_0 + \widetilde{U}_i\widetilde{f}_i)(t,X)\right)\widetilde{f}_i(t,X)\cdot w(X)\,dX, \\ I_i(t) = \int_{\Omega_\epsilon} &W_i(t,X) f_i(t,X)\left(\nabla_X\widetilde{f}_i(t,X) + \frac{X}{2}\widetilde{f}_i(t,X)\right)\cdot w(X)\,dX. \end{aligned}$$

The main result of this subsection states as follows:

Proposition 3.1. *There exists some positive constant δ depending on the initial measure μ, such that for t sufficiently small, there holds*

$$tE_i'(t) \leq -2\delta\mathcal{E}_i(t) + C_0\sqrt{t}|\ln t|\mathcal{E}_i(t)^{\frac{1}{2}} + CE_i(t)^{\frac{1}{2}}\mathcal{E}_i(t) + \mathcal{R}_i(t), \tag{3.5}$$

where the quantity $\mathcal{R}_i$ satisfies the inequality $0 < \mathcal{R}_i(t) \leq e^{-C_0/t}$.

Proof. Noting that the terms in $A_i(t)$ are exactly the same as the ones appearing on the right-hand side of the equality (4.42) in [10]. Thus using the Proposition 4.5 in [10], we know that there exists some $\epsilon_0 \in]0, 1/2[$, if $t > 0$ is small enough so that $\epsilon_i < \epsilon_0$, then

$$A_i(t) \leq -2\delta\mathcal{E}_i(t) + C\sqrt{t}|\ln t|E_i(t)^{\frac{1}{2}} + CE_i(t)^{\frac{1}{2}}\mathcal{E}_i(t) + \mathcal{R}_i(t). \tag{3.6}$$

In the following we shall concentrate on the interaction part $I_i(t)$. Using the decomposition (2.47) and (2.49), we can write

$$W_i(t,X)f_i(t,X) = \sum_{j\neq i}\big(\alpha_j U_{0,j}(t,X_j) + \alpha_j\widetilde{U}_j(t,X_j)\big)\big(f_0(t,X) + \widetilde{f}_i(t,X)\big).$$

Thus there are two types of integral terms in $I_i(t)$, which we handle separately.

Before proceeding, let us decompose Ω_{ϵ_j} into two parts, namely

$$\Omega_{\epsilon_j}^+ \overset{\text{def}}{=} \Big\{X \in \Omega_{\epsilon_j} \text{ s.t. } |X| > \frac{d}{4\sqrt{t}}\Big\}, \quad \Omega_{\epsilon_j}^- \overset{\text{def}}{=} \Big\{X \in \Omega_{\epsilon_j} \text{ s.t. } |X| \leq \frac{d}{4\sqrt{t}}\Big\}.$$

Type 1: $I_{i,1}(t) = \sum_{j\neq i}\int_{\Omega_\epsilon} \alpha_j U_j(t,X_j) f_0(t,X)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i(t,X)\cdot w(X)\,dX.$

Due to the cutoff function χ, we know that $f_0(t,X)$ vanishes whenever $|X| > \frac{d}{4\sqrt{t}}$. Thus $I_{i,1}(t)$ actually only integrates on Ω_ϵ^-, and for X in Ω_ϵ^-, we have

$$|X_j| = \Big|X + \frac{x_i - x_j}{\sqrt{t}}\Big| \geq \frac{3d}{4\sqrt{t}}.$$

Then the estimate (2.38) gives

$$U_j(t,X_j) \leq C_0\sqrt{t}. \tag{3.7}$$

Thanks to this bound, the definition of f_0, and Cauchy inequality, we get

$$\begin{aligned}
|I_{i,1}(t)| &\leq C_0\sqrt{t}\sum_{j\neq i}\int_{\Omega_\epsilon^-} e^{-|X|^2/4}\left(\nabla_X\widetilde{f}_i(t,X) + \frac{X}{2}\widetilde{f}_i(t,X)\right)w(X)\,dX\\
&\leq C_0\sqrt{t}\left\|e^{-|X|^2/8}\right\|_{L^2(\Omega_\epsilon^-)}\left\|(\nabla_X + X/2)\,\widetilde{f}_i(t,X)\cdot w(X)^{1/2}\right\|_{L^2(\Omega_\epsilon^-)}\\
&\leq C_0\sqrt{t}\mathcal{E}_i(t)^{\frac{1}{2}}.
\end{aligned} \tag{3.8}$$

Type 2: $I_{i,2}(t) = \sum_{j\neq i}\int_{\Omega_\epsilon} \alpha_j U_j(t,X_j)\widetilde{f}_i(t,X)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i(t,X)\cdot w(X)\,dX.$

We decompose $I_{i,2}$ into two different parts according to the integral domain. On Ω_ϵ^-, by using the bound (3.7) and Cauchy inequality again, we obtain

$$\Big|\int_{\Omega_\epsilon^-} U_j(t,X_j)\widetilde{f}_i(t,X)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i(t,X)\cdot w(X)\,dX\Big| \leq C_0\sqrt{t}E_i(t)^{\frac{1}{2}}\mathcal{E}_i(t)^{\frac{1}{2}}. \tag{3.9}$$

To handle the integral on Ω_ϵ^+, a mere application of (2.38) gives

$$\|U_j\|_{L^\infty_T(L^\infty(\Omega_{\epsilon_j}))} \leq C_0. \tag{3.10}$$

And it follows from the Gaussian bound for f_i (2.36) and the fact that f_0 vanishes on Ω_ϵ^+ that, the same Gaussian bound also holds for $\widetilde{f}_i$, precisely

$$0 < \widetilde{f}_i(t,X) \leq C_{\eta,\alpha}e^{-\frac{1-\eta}{4}|X|^2}, \quad \forall X \in \Omega_\epsilon^+. \tag{3.11}$$

Using the above bounds (3.10) and (3.11) with $\eta = \frac{1}{4}$, we get

$$\Big|\int_{\Omega_\epsilon^+} U_j(t,X_j)\widetilde{f}_i(t,X)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i(t,X)\cdot w(X)\,dX\Big| \le C_0\|\widetilde{f}_i(t)w^{\frac12}\|_{L^2(\Omega_\epsilon^+)}\mathcal{E}_i(t)^{\frac12}$$
$$\le C_0 e^{-\frac{d^2}{256t}}\mathcal{E}_i(t)^{\frac12}.$$

Combining this with the estimate (3.9), we finally get

$$|I_{i,2}(t)| \le C_0\sqrt{t}E_i(t)^{\frac12}\mathcal{E}_i(t)^{\frac12} + C_0 e^{-\frac{d^2}{256t}}\mathcal{E}_i(t)^{\frac12}. \tag{3.12}$$

Substituting the estimates (3.6), (3.8), (3.12) and using the trivial bounds

$$E_i \le \mathcal{E}_i \le \mathcal{E}, \quad E_i \le E$$

allows us to obtain

$$tE_i'(t) \le -2\delta\mathcal{E}_i(t) + C\sqrt{t}|\ln t|E_i(t)^{\frac12} + CE_i(t)^{\frac12}\mathcal{E}_i(t) + \mathcal{R}_i(t)$$
$$+ C_0\sqrt{t}\mathcal{E}_i(t)^{\frac12} + C_0\sqrt{t}E_i(t)^{\frac12}\mathcal{E}_i(t)^{\frac12} + C_0 e^{-\frac{d^2}{256t}}\mathcal{E}_i(t)^{\frac12}.$$

Recalling that $E(t)$ goes to 0 as t goes to 0 yields the simplified bound

$$tE_i'(t) \le -2\delta\mathcal{E}_i(t) + C_0\sqrt{t}|\ln t|\mathcal{E}_i(t)^{\frac12} + CE_i(t)^{\frac12}\mathcal{E}_i(t) + \mathcal{R}_i(t),$$

which is the desired differential inequality. This completes the proof of this proposition. □

Proof of the estimate (1.5). Applying Young's inequality to (3.5) gives

$$tE_i'(t) \le -\frac{3}{2}\delta\mathcal{E}_i(t) + C_0 t|\ln t|^2 + CE_i(t)^{\frac12}\mathcal{E}_i(t) + \mathcal{R}_i(t). \tag{3.13}$$

Recalling that by definition $\epsilon_i = \sqrt{t}/r_i$ and $E(t)$ goes to 0 as t goes to 0, thus there exists some small constant t_0 depending only on the initial measure μ, such that both $\epsilon_i < \epsilon_0$ and $E_i(t)^{1/2} < \delta/2$ hold whenever $t < t_0$. Combining this with the facts that $E_i \le \mathcal{E}_i$ and $0 < \mathcal{R}_i(t) \le e^{-C_0/t}$, we can get from (3.13), for $t < t_0$, that

$$tE_i'(t) \le -\delta\mathcal{E}_i(t) + C_0 t|\ln t|^2 + \mathcal{R}_i(t)$$
$$\le -\delta E_i(t) + C_0 t|\ln t|^2.$$

Integrating this differential inequality yields the bound

$$E_i(t) \le C_0 t^{-\delta}\int_0^t s^\delta|\ln s|^2\,ds \le C_0 t|\ln t|^2. \tag{3.14}$$

Then in view of the definition (3.3), the above inequality leads to

$$\|f_i(t) - f_0(t)\|_{L^1(\Omega_\epsilon)} = \|\widetilde{f}_i\|_{L^1(\Omega_\epsilon)} \le CE_i^{1/2}(t) \le C_0\sqrt{t}|\ln t|.$$

And since f_0 is extremely close to G, we finally obtain

$$\begin{aligned}\|f_i(t) - G\|_{L^1(\Omega_\epsilon)} &\le \|f_i(t) - f_0(t)\|_{L^1(\Omega_\epsilon)} + \|f_0(t) - G\|_{L^1(\Omega_\epsilon)}\\ &\le C_0\sqrt{t}|\ln t| + e^{-C_0/t} \le C_0\sqrt{t}|\ln t|.\end{aligned} \tag{3.15}$$

Returning to the original variables, and summing up over i, gives exactly the short time estimate (1.5) for $t < t_0$. □

3.2. Uniqueness

The purpose of this final subsection is to prove the uniqueness result in Theorem 1.1. Assume that $\omega^{\theta,(1)}$, $\omega^{\theta,(2)} \in \mathcal{C}\big(]0,T[, L^1(\Omega)\cap L^\infty(\Omega)\big)$ are two mild solutions to the vorticity Eq. (1.2) satisfying (1.4). Introducing the self-similar variables and decompose these two solutions just as what we have done in Subsection 2.2, precisely for $\ell = 1, 2$, we write

$$\omega^{\theta,(\ell)}(t,r,z) = \sum_{j=1}^{n} \frac{\alpha_j}{t} f_j^{(\ell)}(t,R_j,Z_j) = \sum_{j=1}^{n}\Big(\frac{\alpha_j}{t} f_0(t,R_j,Z_j) + \frac{\alpha_j}{t}\widetilde{f}_j^{(\ell)}(t,R_j,Z_j)\Big),$$

and correspondingly, $u^{(\ell)} = BS[\omega^{\theta,(\ell)}]$ can be decomposed into

$$u(t,r,z)^{(\ell)} = \sum_{j=1}^{n} \frac{\alpha_j}{\sqrt{t}} U_j^{(\ell)}(t,R_j,Z_j) = \sum_{j=1}^{n}\Big(\frac{\alpha_j}{\sqrt{t}} U_{0,j}(t,R_j,Z_j) + \frac{\alpha_j}{\sqrt{t}}\widetilde{U}_j^{(\ell)}(t,R_j,Z_j)\Big).$$

The differences of the rescaled solutions will be denoted by

$$\widetilde{f}_i^{\Delta} \overset{\text{def}}{=} f_i^{(1)} - f_i^{(2)} = \widetilde{f}_i^{(1)} - \widetilde{f}_i^{(2)}, \quad \widetilde{U}_i^{\Delta} \overset{\text{def}}{=} U_i^{(1)} - U_i^{(2)} = \widetilde{U}_i^{(1)} - \widetilde{U}_i^{(2)}.$$

The evolution equation for $\widetilde{f}_i^{\Delta}$ reads

$$\begin{aligned} &t\partial_t \widetilde{f}_i^{\Delta} + \alpha_i \operatorname{div}_*(U_{0,i}\widetilde{f}_i^{\Delta} + \widetilde{U}_i^{\Delta} f_0) + \alpha_i \operatorname{div}_*(\widetilde{U}_i^{(1)}\widetilde{f}_i^{(1)} - \widetilde{U}_i^{(2)}\widetilde{f}_i^{(2)}) \\ &\quad + \operatorname{div}_*(W_{0,i}\widetilde{f}_i^{\Delta} + \widetilde{W}_i^{\Delta} f_0) + \operatorname{div}_*(\widetilde{W}_i^{(1)}\widetilde{f}_i^{(1)} - \widetilde{W}_i^{(2)}\widetilde{f}_i^{(2)}) = \mathcal{L}\widetilde{f}_i^{\Delta} + \partial_R\Big(\frac{\epsilon \widetilde{f}_i^{\Delta}}{1+\epsilon R}\Big), \end{aligned} \tag{3.16}$$

where

$$W_{0,i}(t,X) \overset{\text{def}}{=} \sum_{j\neq i} \alpha_j U_{0,j}(t,X_j), \quad \widetilde{W}_i^{(\ell)}(t,X) \overset{\text{def}}{=} \sum_{j\neq i} \alpha_j \widetilde{U}_j^{(\ell)}(t,X_j).$$

In analogy with (3.3), the energies for each solution are straightforwardly denoted by

$$\begin{aligned} &E_j^{(\ell)}(t) \overset{\text{def}}{=} \frac{1}{2}\int_{\Omega_{\epsilon_j}} \widetilde{f}_j^{(\ell)}(t,X_j)^2 w(X_j)\,dX_j, \quad E^{(\ell)}(t) \overset{\text{def}}{=} \sum_{j=1}^{n} E_j^{(\ell)}(t), \\ &\mathcal{E}_j^{(\ell)}(t) \overset{\text{def}}{=} \frac{1}{2}\int_{\Omega_{\epsilon_j}} \Big(|\nabla \widetilde{f}_j^{(\ell)}(t,X_j)|^2 + (1+|X_j|^2)\widetilde{f}_j^{(\ell)}(t,X_j)^2\Big) w(X_j)\,dX_j, \ \mathcal{E}^{(\ell)}(t) \overset{\text{def}}{=} \sum_{j=1}^{n} \mathcal{E}_j^{(\ell)}(t), \end{aligned}$$

as well as the energies for the difference

$$\begin{aligned} &E_j^{\Delta}(t) \overset{\text{def}}{=} \frac{1}{2}\int_{\Omega_{\epsilon_j}} \widetilde{f}_j^{\Delta}(t,X_j)^2 w(X_j)\,dX_j, \quad E^{\Delta}(t) \overset{\text{def}}{=} \sum_{j=1}^{n} E_j^{\Delta}(t), \\ &\mathcal{E}_j^{\Delta}(t) \overset{\text{def}}{=} \frac{1}{2}\int_{\Omega_{\epsilon_j}} \Big(|\nabla \widetilde{f}_j^{\Delta}(t,X_j)|^2 + (1+|X_j|^2)\widetilde{f}_j^{\Delta}(t,X_j)^2\Big) w(X_j)\,dX_j, \quad \mathcal{E}^{\Delta}(t) \overset{\text{def}}{=} \sum_{j=1}^{n} \mathcal{E}_j^{\Delta}(t). \end{aligned}$$

In view of (3.14), combining with the elementary fact that $E_j^{\Delta} \le 2\big(E_j^{(1)} + E_j^{(2)}\big)$, we know that $E_j^{\Delta}(t)$ also decays to 0 with rate at least $t|\ln t|^2$ as $t \to 0$. We believe that $E_j^{\Delta}(t)$ decays faster than $E_j^{(\ell)}$ since the source $\mathcal{H}$ and $\operatorname{div}_*(W_{0,i} f_0)$ has disappeared when taking the difference of the equations for $f_i^{(1)}$ and $f_i^{(2)}$. Precisely, we have:

Proposition 3.2. *There exists a positive time t_1 such that for all $0 < t < t_1$, there holds*

$$E^{\Delta}(t) \le e^{-C_0/t}. \tag{3.17}$$

Proof. Similarly as in the proof of Proposition 3.1, by doing an $L^2(\Omega_\epsilon, w(X)dX)$ energy estimate to (3.16) and integrating by parts, we obtain

$$t\frac{d}{dt}E_i^\Delta(t) = A_i^\Delta(t) + I_i^\Delta(t), \tag{3.18}$$

where

$$\begin{aligned} A_i^\Delta(t) = \int_{\Omega_\epsilon} \Bigg(& \mathcal{L}\widetilde{f}_i^\Delta(t,X) + \partial_R\left(\frac{\epsilon \widetilde{f}_i^\Delta(t,X)}{1+\epsilon R}\right) - \alpha_i \operatorname{div}_*(U_{0,i}\widetilde{f}_i^\Delta + \widetilde{U}_i^\Delta f_0) \\ & -\alpha_i \operatorname{div}_*(\widetilde{U}_i^{(1)}\widetilde{f}_i^{(1)} - \widetilde{U}_i^{(2)}\widetilde{f}_i^{(2)})\Bigg)\widetilde{f}_i^\Delta(t,X)\cdot w(X)\,dX, \\ I_i^\Delta(t) = \int_{\Omega_\epsilon} & \left(W_{0,i}\widetilde{f}_i^\Delta + \widetilde{W}_i^\Delta f_0 + \widetilde{W}_i^{(1)}\widetilde{f}_i^{(1)} - \widetilde{W}_i^{(2)}\widetilde{f}_i^{(2)}\right)(t,X)\cdot(\nabla_X + X/2)\,\widetilde{f}_i^\Delta(t,X)\cdot w(X)\,dX. \end{aligned}$$

First, the estimate (4.71) of [10] claims that there exists some positive constant δ and some $\epsilon_0 \in]0,1[$ such that as long as $\epsilon < \epsilon_0$, there holds

$$A_i^\Delta(t) \le -2\delta\mathcal{E}_i^\Delta(t) + C\big(E_i^{(1)}(t)^{\frac12} + E_i^{(2)}(t)^{\frac12}\big)\mathcal{E}_i^\Delta(t) + \mathcal{R}_i^\Delta(t), \tag{3.19}$$

where the quantity $\mathcal{R}_i^\Delta$ satisfies the inequality $0 < \mathcal{R}_i^\Delta(t) \le e^{-C_0/t}$. We mention that the terms with type $C_0\sqrt{t}|\ln t|\mathcal{E}_i(t)^{\frac12}$ in (3.6) does not appear here, due to the cancellation of the source term $\mathcal{H}$ when taking the difference.

For the interaction part $I_i^\Delta(t)$, thanks to the cancellation of $\operatorname{div}_*(W_{0,i}f_0)$, there are only three types of integral terms, which we handle separately in the following.

Type 1: $I_{i,1}^\Delta(t) = \int_{\Omega_\epsilon} W_{0,i}(t,X)\widetilde{f}_i^\Delta(t,X)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i^\Delta(t,X)\cdot w(X)\,dX.$

We decompose $I_{i,1}^\Delta$ into two different parts according to the integral domain. On Ω_ϵ^-, we have the pointwise estimate:

Lemma 3.1. *For any $j \neq i$, and any X_j in $\Omega_{\epsilon_j}^-$ (i.e. X in Ω_ϵ^-), we have*

$$|U_{0,j}(t,X_j)| \le C_0\sqrt{t}.$$

Proof. Using the explicit formula (2.34), and the fact that f_0 is supported inside Ω_ϵ^-, we get

$$\begin{aligned} U_{0,j}^r(t,X_j) &= \frac{1}{2\pi}\int_{\Omega_\epsilon^-}\sqrt{(1+\epsilon_j R')(1+\epsilon_j R_j)^{-1}}F_1(\xi_j^2)\frac{Z_j - Z'}{|X_j - X'|^2}f_0(t,X')\,dX', \\ U_{0,j}^z(t,X_j) &= -\frac{1}{2\pi}\int_{\Omega_\epsilon^-}\sqrt{(1+\epsilon_j R')(1+\epsilon_j R_j)^{-1}}F_1(\xi_j^2)\frac{R_j - R'}{|X_j - X'|^2}f_0(t,X')\,dX' \\ &\quad + \frac{\epsilon_j}{4\pi}\int_{\Omega_\epsilon^-}\sqrt{(1+\epsilon_j R')(1+\epsilon_j R_j)^{-3}}\big(F_1(\xi_j^2) + F_2(\xi_j^2)\big)f_0(t,X')\,dX', \end{aligned}$$

where

$$\xi_j^2 = \epsilon_j^2|X_j - X'|^2(1+\epsilon_j R_j)^{-1}(1+\epsilon_j R')^{-1}.$$

For X and X' in Ω_ϵ^-, we have

$$\begin{aligned} |X_j - X'| &= \left|X - X' + \frac{x_i - x_j}{\sqrt{t}}\right| \in \left[\frac{d}{2\sqrt{t}}, \frac{d + 2|x_i - x_j|}{2\sqrt{t}}\right], \\ 1 + \epsilon_j R' &\in \left[\frac34, \frac54\right], \quad \text{and} \quad 1 + \epsilon_j R_j = \frac{r_i}{r_j} + \frac{\sqrt{t}R}{r_j} \in \left[\frac{3r_i}{4r_j}, \frac{5r_i}{4r_j}\right]. \end{aligned}$$

Using the above bounds and the fact that $F_1(s)$, $s^{\frac12}F_2(s)$ are bounded on $]0,\infty[$, we achieve

$$|U_{0,j}(X_j)| \le C_0\int_{\Omega_\epsilon^-}\sqrt{t}e^{-|X'|^2/4}\,dX' \le C_0\sqrt{t},$$

which completes the proof of this lemma. □

A direct consequence of this lemma is that, $W_{0,i}(t,X) \le C_0\sqrt{t}$ for any $X \in \Omega_\epsilon^-$. Using this pointwise bound and Cauchy inequality, we obtain

$$\Big|\int_{\Omega_\epsilon^-} W_{0,i}(t,X)\widetilde{f}_i^\Delta(t,X)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i^\Delta(t,X)\cdot w(X)\,dX\Big| \le C_0\sqrt{t}E_i^\Delta(t)^{\frac12}\mathcal{E}_i^\Delta(t)^{\frac12}. \tag{3.20}$$

To handle the integral on Ω_ϵ^+, we need some more careful estimates on the rescaled velocity. After the blow-up procedure (2.33), Proposition 2.3 of [9] translates into:

Lemma 3.2. (i) *If* $1 < p < 2 < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, *then*

$$\|BS^\epsilon[f]\|_{L^q(\Omega_\epsilon)} \le C\|f\|_{L^p(\Omega_\epsilon)}. \tag{3.21}$$

(ii) *If* $1 \le p < 2 < q \le \infty$, *then*

$$\|BS^\epsilon[f]\|_{L^\infty(\Omega_\epsilon)} \le C\|f\|^\sigma_{L^p(\Omega_\epsilon)}\|f\|^{1-\sigma}_{L^q(\Omega_\epsilon)}, \quad \textit{where} \quad \sigma = \frac{p}{2}\frac{q-2}{q-p} \in]0,1[. \tag{3.22}$$

It follows from a mere application of (3.22) to a gaussian function that

$$\|W_{0,i}\|_{L^\infty_T(L^\infty(\Omega_\epsilon))} \le C\|f_0\|^{\frac12}_{L^1}\|f_0\|^{\frac12}_{L^\infty} \le C. \tag{3.23}$$

And it follows from the Gaussian bound for $f_i^{(\ell)}$ (2.36) and the fact that f_0 vanishes on Ω_ϵ^+ that, the same Gaussian bound also holds for $\widetilde{f}_i^{(\ell)}$, precisely

$$0 < \widetilde{f}_i^{(\ell)}(t,X) \le C_{\eta,\alpha}e^{-\frac{1-\eta}{4}|X|^2}, \quad \forall X \in \Omega_\epsilon^+. \tag{3.24}$$

Using the above bounds (3.23) and (3.24) with $\eta = \frac14$, we get

$$\begin{aligned}\Big|\int_{\Omega_\epsilon^+} W_{0,i}(t,X)\widetilde{f}_i^\Delta(t,X)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i^\Delta(t,X)\cdot w(X)\,dX\Big| &\le C\|\widetilde{f}_i^\Delta(t)w^{\frac12}\|_{L^2(\Omega_\epsilon^+)}\mathcal{E}_i^\Delta(t)^{\frac12}\\ &\le C_0e^{-\frac{d^2}{256t}}\mathcal{E}_i^\Delta(t)^{\frac12}.\end{aligned}$$

Combining this with the estimate (3.20), we finally get

$$|I_{i,1}^\Delta(t)| \le C_0\sqrt{t}E_i^\Delta(t)^{\frac12}\mathcal{E}_i^\Delta(t)^{\frac12} + C_0e^{-\frac{d^2}{256t}}\mathcal{E}_i^\Delta(t)^{\frac12}. \tag{3.25}$$

Type 2: $I_{i,2}^\Delta(t) = \int_{\Omega_\epsilon}\widetilde{W}_i^\Delta(t,X)f_0(t,X)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i^\Delta(t,X)\cdot w(X)\,dX$.

Noting that f_0 supports only on Ω_ϵ^-, and $f_0(X)w(X) \le 1$ on Ω_ϵ, we get

$$|I_{i,2}^\Delta(t)| \le \int_{\Omega_\epsilon^-}\sum_{j\neq i}\big|\alpha_j(\widetilde{U}_j^{(1)} - \widetilde{U}_j^{(2)})(t,X_j)\cdot\big(\nabla_X + X/2\big)\widetilde{f}_i^\Delta(t,X)\big|\,dX. \tag{3.26}$$

Let us decompose $\widetilde{U}_j^{(\ell)}$ as the sum of $\widetilde{U}_j^{(\ell),+}$ and $\widetilde{U}_j^{(\ell),-}$, with

$$\widetilde{U}_j^{(\ell),\pm}(X_j) \stackrel{\text{def}}{=} BS^{\epsilon_j}\big[\widetilde{f}_j^{(\ell)}(X_j)\mathbf{1}_{\Omega^\pm_{\epsilon_j}}(X_j)\big],$$

where $\mathbf{1}_{\Omega_\epsilon^\pm}$ stands for the characteristic function of $\Omega_\epsilon^\pm$.

Exactly along the proof of Lemma 3.1, we can get, for any $X \in \Omega_\epsilon^-$, that

$$\begin{aligned}\big|\big(\widetilde{U}_j^{(1),-} - \widetilde{U}_j^{(2),-}\big)\big(X + \frac{x_i - x_j}{\sqrt{t}}\big)\big| &\le C_0\sqrt{t}\int_{\Omega^-_{\epsilon_j}}\big|\widetilde{f}_j^{(1)}(X') - \widetilde{f}_j^{(2)}(X')\big|\,dX'\\ &\le C_0\sqrt{t}\|w^{-1/2}\|_{L^2}E_j^\Delta(t)^{\frac12}\\ &\le C_0\sqrt{t}E_j^\Delta(t)^{\frac12}.\end{aligned}$$

Using this bound and the fact that $L^2\big(\Omega_\epsilon^-, w(X)dX\big) \hookrightarrow L^1(\Omega_\epsilon^-, dX)$, we achieve

$$\begin{aligned} &\int_{\Omega_\epsilon^-} \sum_{j\neq i} \big|\alpha_j (\widetilde{U}_j^{(1),-} - \widetilde{U}_j^{(2),-})(t, X_j) \cdot \big(\nabla_X + X/2\big) \widetilde{f}_i^{\Delta}(t, X)\big|\, dX \\ &\qquad \le C_0 \sqrt{t} E^{\Delta}(t)^{\frac12} \mathcal{E}_i^{\Delta}(t)^{\frac12}. \end{aligned} \tag{3.27}$$

For $\widetilde{U}_j^{(\ell),+}$, we use (3.21) with $p = 4/3,\ q = 4$, and Hölder's inequality to obtain

$$\begin{aligned} \big\|\widetilde{U}_j^{(1),+} - \widetilde{U}_j^{(2),+}\big\|_{L^4(\Omega_{\epsilon_j})} &\le C_0 \big\|\widetilde{f}_j^{(1)} - \widetilde{f}_j^{(2)}\big\|_{L^{\frac43}(\Omega_{\epsilon_j}^+)} \\ &\le C_0 \|w^{-1/2}\|_{L^4(\Omega_{\epsilon_j}^+)} \big\|\big(\widetilde{f}_j^{(1)} - \widetilde{f}_j^{(2)}\big) w^{1/2}\big\|_{L^2(\Omega_{\epsilon_j}^+)} \\ &\le C_0 e^{-C_0/t} E_j^{\Delta}(t)^{\frac12}. \end{aligned}$$

Using this estimate and Hölder's inequality again, we achieve

$$\begin{aligned} &\int_{\Omega_\epsilon^-} \sum_{j\neq i} \big|\alpha_j (\widetilde{U}_j^{(1),+} - \widetilde{U}_j^{(2),+})(t, X_j) \cdot \big(\nabla_X + X/2\big) \widetilde{f}_i^{\Delta}(t, X)\big|\, dX \\ &\quad \le \sum_{j\neq i} \big\|\widetilde{U}_j^{(1),+} - \widetilde{U}_j^{(2),+}\big\|_{L^4(\Omega_\epsilon^-)} \|w^{-1/2}\|_{L^4(\Omega_\epsilon^-)} \big\|\big(\nabla_X + X/2\big) \widetilde{f}_i^{\Delta} \cdot w^{1/2}\big\|_{L^2(\Omega_\epsilon^-)} \\ &\quad \le C_0 e^{-C_0/t} E^{\Delta}(t)^{\frac12} \mathcal{E}_i^{\Delta}(t)^{\frac12}. \end{aligned} \tag{3.28}$$

Combining the estimates (3.27) and (3.28), we finally achieve that

$$|I_{i,2}^{\Delta}(t)| \le C_0 \sqrt{t} E^{\Delta}(t)^{\frac12} \mathcal{E}_i^{\Delta}(t)^{\frac12}. \tag{3.29}$$

Type 3: $I_{i,3}^{\Delta}(t) = \int_{\Omega_\epsilon} \big(\widetilde{W}_i^{(1)} \widetilde{f}_i^{(1)} - \widetilde{W}_i^{(2)} \widetilde{f}_i^{(2)}\big)(t, X) \cdot \big(\nabla_X + X/2\big) \widetilde{f}_i^{\Delta}(t, X) \cdot w(X)\, dX.$

The strategy of estimating $I_{i,3}^{\Delta}(t)$ is to write

$$\widetilde{W}_i^{(1)} \widetilde{f}_i^{(1)} - \widetilde{W}_i^{(2)} \widetilde{f}_i^{(2)} = \widetilde{W}_i^{\Delta} \widetilde{f}_i^{(1)} + \widetilde{W}_i^{(2)} \widetilde{f}_i^{\Delta},$$

where $\widetilde{W}_i^{\Delta} \stackrel{\text{def}}{=} \widetilde{W}_i^{(1)} - \widetilde{W}_i^{(2)}$. Then we get, by using Hölder's inequality, that

$$\begin{aligned} |I_{i,3}^{\Delta}(t)| &\le \Big(\|\widetilde{W}_i^{\Delta}\|_{L^\infty(\Omega_\epsilon)} \Big\|\widetilde{f}_i^{(1)} w^{\frac12}\Big\|_{L^2(\Omega_\epsilon)} + \|\widetilde{W}_i^{(2)}\|_{L^\infty(\Omega_\epsilon)} \|\widetilde{f}_i^{\Delta} w^{\frac12}\|_{L^2(\Omega_\epsilon)} \Big) \\ &\qquad \times \big\|\big(\nabla_X + X/2\big) \widetilde{f}_i^{\Delta} w^{\frac12}\big\|_{L^2(\Omega_\epsilon)} \\ &\le \Big(\|\widetilde{W}_i^{\Delta}\|_{L^\infty(\Omega_\epsilon)} E_i^{(1)}(t)^{\frac12} + \|\widetilde{W}_i^{(2)}\|_{L^\infty(\Omega_\epsilon)} E_i^{\Delta}(t)^{\frac12} \Big) \mathcal{E}_i^{\Delta}(t)^{\frac12}. \end{aligned} \tag{3.30}$$

By using (3.22) with $p = 4/3,\ q = 4$, and Gagliardo-Nirenberg-Ladyzhenskaya inequality, we obtain

$$\begin{aligned} \|\widetilde{W}_i^{\Delta}\|_{L^\infty(\Omega_\epsilon)} &\le C_0 \sum_{j\neq i} \|\widetilde{f}_j^{\Delta}\|_{L^{4/3}(\Omega_\epsilon)}^{1/2} \|\widetilde{f}_j^{\Delta}\|_{L^4(\Omega_\epsilon)}^{1/2} \\ &\le C_0 \sum_{j\neq i} \|\widetilde{f}_j^{\Delta} w^{1/2}\|_{L^2(\Omega_\epsilon)}^{1/2} \|w^{-1/2}\|_{L^2(\Omega_\epsilon)}^{1/2} \|\widetilde{f}_j^{\Delta}\|_{L^2(\Omega_\epsilon)}^{1/4} \|\nabla \widetilde{f}_j^{\Delta}\|_{L^2(\Omega_\epsilon)}^{1/4} \\ &\le C_0 \sum_{j\neq i} E_j^{\Delta}(t)^{\frac38} \mathcal{E}_j^{\Delta}(t)^{\frac18}. \end{aligned}$$

Similarly, and noting that $\widetilde{f}_j^{(2)}$ satisfies the pointwise estimate (3.24), we obtain

$$\begin{aligned} \|\widetilde{W}_i^{(2)}\|_{L^\infty(\Omega_\epsilon)} &\le C_0 \sum_{j\neq i} \|\widetilde{f}_j^{(2)}\|_{L^{4/3}(\Omega_\epsilon)}^{1/2} \|\widetilde{f}_j^{(2)}\|_{L^4(\Omega_\epsilon)}^{1/2} \\ &\le C_0 \sum_{j\neq i} E_j^{(2)}(t)^{\frac14}. \end{aligned}$$

Substituting the above two estimates into (3.30), we achieve

$$|I_{i,3}^{\Delta}(t)| \le C_0\Big(E_i^{(1)}(t)^{\frac{1}{2}}E^{\Delta}(t)^{\frac{3}{8}}\mathcal{E}^{\Delta}(t)^{\frac{1}{8}} + E^{(2)}(t)^{\frac{1}{4}}E_i^{\Delta}(t)^{\frac{1}{2}}\Big)\mathcal{E}_i^{\Delta}(t)^{\frac{1}{2}}. \tag{3.31}$$

Overall, by putting (3.25), (3.29) and (3.31) together, using Young's inequality and the fact that $E_i^{\Delta} \le \mathcal{E}_i^{\Delta} \le \mathcal{E}^{\Delta}$, we achieve

$$I^{\Delta}(t) \le \delta\mathcal{E}_i^{\Delta}(t) + C_0\big(\sqrt{t} + E_i^{(1)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{4}}\big)\mathcal{E}^{\Delta}(t) + C_0e^{-C_0/t}. \tag{3.32}$$

Then substituting (3.19) and (3.32) into (3.18), and summing up over i, leads to

$$t\frac{d}{dt}E^{\Delta}(t) \le -\delta\mathcal{E}^{\Delta}(t) + C_0\big(\sqrt{t} + E^{(1)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{4}}\big)\mathcal{E}^{\Delta}(t) + C_0e^{-C_0/t}. \tag{3.33}$$

The bound (3.14) guarantees the existence of a positive time t_1, such that for all $0 < t < t_1$, there holds $C_0\big(\sqrt{t} + E^{(1)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{4}}\big) \le \frac{\delta}{2}$. Then (3.33) turns into

$$t\frac{d}{dt}E^{\Delta}(t) \le -\frac{\delta}{2}\mathcal{E}^{\Delta}(t) + C_0e^{-C_0/t} \le -\frac{\delta}{2}E^{\Delta}(t) + C_0e^{-C_0/t}. \tag{3.34}$$

Then integrating this differential inequality from 0 to $t < t_1$ gives

$$E^{\Delta}(t) \le C_0t^{-\delta/2}\int_0^t s^{\delta/2-1}e^{-C_0/s}\,ds \le e^{-C_0/t},$$

which is exactly the desired estimate (3.17). □

Proposition 3.2 already shows that $E^{\Delta}(t)$ converges extremely rapidly to 0 as $t \to 0$, but our actual goal is to prove that $E^{\Delta}(t)$ vanishes identically, which will be done in the following.

Proof of the uniqueness result in Theorem 1.1. The key is to get a new differential inequality for $E^{\Delta}(t)$ like (3.34), but in which the "inhomogeneous" term like $C_0e^{-C_0/t}$ does not appear.

First, the estimate (4.73) of [10] claims that as long as $\epsilon < 1/2$, there holds

$$A_i^{\Delta}(t) \le -\delta\mathcal{E}_i^{\Delta}(t) + C_0E_i^{\Delta}(t) + C_0\big(E_i^{(1)}(t)^{\frac{1}{2}} + E_i^{(2)}(t)^{\frac{1}{2}}\big)\mathcal{E}_i^{\Delta}(t). \tag{3.35}$$

For the estimate of $I_i^{\Delta}(t)$, we only need to modify the estimate of $I_{i,1}^{\Delta}(t)$. By simply using the bound for U_i given by (2.38), we can achieve

$$|I_{i,1}^{\Delta}(t)| \le C_0E_i^{\Delta}(t)^{\frac{1}{2}}\mathcal{E}_i^{\Delta}(t)^{\frac{1}{2}}.$$

The other terms in $I_i^{\Delta}(t)$ can be estimated exactly along the proof of Proposition 3.2. Then for small t, we deduce

$$\begin{aligned}|I_i^{\Delta}(t)| &\le C_0E^{\Delta}(t)^{\frac{1}{2}}\mathcal{E}^{\Delta}(t)^{\frac{1}{2}} + C_0\big(E_i^{(1)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{4}}\big)\mathcal{E}^{\Delta}(t)\\ &\le \frac{\delta}{2n}\mathcal{E}^{\Delta}(t) + C_0E^{\Delta}(t) + C_0\big(E_i^{(1)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{4}}\big)\mathcal{E}^{\Delta}(t).\end{aligned} \tag{3.36}$$

Substituting (3.35) and (3.36) into (3.18), and summing up over i, leads to

$$t\frac{d}{dt}E^{\Delta}(t) \le -\frac{\delta}{2}\mathcal{E}^{\Delta}(t) + C_0E^{\Delta}(t) + C_0\big(E^{(1)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{4}}\big)\mathcal{E}^{\Delta}(t). \tag{3.37}$$

The bound (3.14) guarantees the existence of a positive time t_2, such that for all $0 < t < t_2$, there holds $C_0\big(\sqrt{t} + E^{(1)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{2}} + E^{(2)}(t)^{\frac{1}{4}}\big) \le \frac{\delta}{2}$. Then (3.37) turns into

$$t\frac{d}{dt}E^{\Delta}(t) \le C_0E^{\Delta}(t),$$

hence

$$E^{\Delta}(t) \le \left(\frac{t}{t'}\right)^{C_0}E^{\Delta}(t'), \quad \forall 0 < t' < t. \tag{3.38}$$

In view of (3.17), the right-hand side of (3.38) converges to 0 as $t' \to 0$. Thus $E^{\Delta}(t) = 0$, which means that $f^{(1)}(t) = f^{(2)}(t)$ for all $0 < t < \min(t_1, t_2)$. Returning to the original variables, we conclude that

$\omega^{\theta,(1)}(t) = \omega^{\theta,(2)}(t)$ for all $0 < t < \min(t_1, t_2)$. Then the desired uniqueness follows from the global well-posedness result established in Theorem 1.1 of [9], and the whole theorem has been proved. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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